Metric spaces 8: Gromov hyperbolic groups

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Exercise 8.1. Consider a map $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ taking x to $(x, |x|)$. Prove that γ is quasi-isometry to its image $\gamma(\mathbb{R})$.

a. Prove that the metric on $\gamma(\mathbb{R})$ induced from $\mathbb{R}^2$ is not geodesic.

b. Prove that $\mathbb{R}$ is Gromov hyperbolic, and $\gamma(\mathbb{R})$ is not Gromov hyperbolic.

Remark 8.1. It is true that Gromov hyperbolicity is a quasi-isometry invariant. However, this can be applied only to geodesic metric spaces (or metric spaces with intrinsic metric).

Exercise 8.2. Suppose that the metric space (M, d) satisfies $d(x, y) \leqslant \max(d(x, z), d(y, z))$. Prove that (M, d) is Gromov 0-hyperbolic.

Exercise 8.3. Prove that the group $(\mathbb{Z}/n_1\mathbb{Z}) * (\mathbb{Z}/n_2\mathbb{Z}) * ... * (\mathbb{Z}/n_k\mathbb{Z})$ is Gromov hyperbolic.

Exercise 8.4. A p**-regular tree** is an infinite tree where any vertex is joined by p edges. Prove that all p-regular trees, for all $p > 2$, are quasi-isometric.

Exercise 8.5. Let $x \in \Gamma$ be an element of a group generated by S.

a. Prove that the function $n \mapsto d(1, x^n)$ is subadditive, that is, satisfies $d(1, x^n) + d(1, x^m) \leqslant d(1, x^{n+m})$.

b. Define the **asymptotic translation length** as $\rho(x) := \lim_n \frac{d(1,x^n)}{n}$. Prove that this limit converges.

c. Prove that ρ is conjugation invariant, that is, $\rho(xyx^{-1}) = \rho(y)$.

d. Consider the group $\mathbb{Z}$ with the set of generators $S = \{\pm 1, \pm N\}$. Find an element of $\mathbb{Z}$ which has non-integral translation length.

e. **Baumslag-Solitar group** $BS(m, n)$ is a group generated by x, y, with the only relation $xy^m x^{-1} = y^n$. Find a non-trivial element $w \in BS(1, n)$ which satifies $\rho(w) = 0$.

f. Let $m \neq \pm n$. Find a non-trivial element $w \in BS(m, n)$ which satifies $\rho(w) = 0$, or find a counterexample.

g. Prove that $\rho(w) \neq 0$ for any non-trivial element in a free group.

h. (**) Prove that $\rho(w) \neq 0$ for any element of infinite order in a Gromov hyperbolic group.

Exercise 8.6 (*). Let Γ be a Gromov hyperbolic group. Prove that any group homomorphism $\mathbb{Z}^2 \longrightarrow \Gamma$ has non-zero kernel.