Metric geometry, final exam

Rules: Every student gets 10 exercises, the final grade is determined by the score. The mark is C for score 30-49, B for 50-59, A for 60-79, A+ for the higher score. Please write down the solutions and bring them in person no later than April 2023.

1 Metric spaces

Exercise 1.1 (10 points). Prove that any metric space admits an isometric embedding to a normed vector space.

Exercise 1.2 (20 points). Let $f: M \longrightarrow M$ be a map of a metric compact to itself, possibly not continuous, such that $d(f(x), f(y)) \geqslant d(x, y)$. Prove that f is an isometry.

Exercise 1.3 (10 points). Non-Archimedean inequality is the inequality $d(x, y) \leqslant \max(d(x, z), d(y, z))$. A metric space which satisfies this inequality is called **ultrametric**. Let M be an infinite ultrametric space. Prove that M is disconnected, or find a counterexample.

Exercise 1.4 (20 points). Let M be a compact metric space, and $\phi : M \longrightarrow M$ an isometric map. Prove that ϕ is surjective.

Exercise 1.5 (20 points). Let M be a compact metric space, and $\phi : M \longrightarrow M$ a surjective, 1-Lipschitz map. Prove that ϕ is an isometry.

Exercise 1.6 (20 points). Prove that any compact metric space can be isometrically embedded to a geodesic compact metric space.

Exercise 1.7 (10 points). Prove that any Lipschitz function $\phi : [0, 1] \longrightarrow [0, 1]$ is differentiable almost everywhere.

Exercise 1.8 (10 points). Let $\phi : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be a non-trivial isometry which has no fixed points. Prove that ϕ preserves at most one geodesic. Find an example of an isometry which does not preserve a geodesic.

2 Intrinsic metrics

Exercise 2.1 (10 points). Construct a locally compact geodesic space such that its completion is not geodesic

Definition 2.1. Recall that a metric d on M is called **path metric**, or **intrinsic metric**, for any $\varepsilon > 0$ and any two points $x, y \in M$ there exists a rectifiable path γ connecting x to y such that $d(x, y) + \varepsilon \geqslant L_d(\gamma)$.

Exercise 2.2 (10 points). Let (M, d) be a space with intrinsic metric, and $A \subset M$ a connected open subset. Prove that A admits a unique intrinsic metric d' such that each point $a \in A$ is contained in a neighbourhood $U \subset A$ such that $d\Big|_U = d'\Big|_U$.

Exercise 2.3 (20 points). Let X be a space with intrinsic metric, homeomorphic to a circle. Prove that there is no isometric embeddings from X to a finite-dimensional normed vector space, or find a counterexample.

Exercise 2.4 (10 points). Give an example of a complete metric space (M, d) with intrinsic metrics, such that for some points $x, y \in M$ there is no minimizing geodesic connecting x to y.

Exercise 2.5 (30 points). Give an example of a complete metric space (M, d) with intrinsic metrics, such that no subset of M isometric to an interval $[0, a] \subset \mathbb{R}$.

Exercise 2.6 (20 points). Let V be a normed vector space. Prove that either the norm on V is Euclidean, or V contains two rays v_1, v_2 starting in 0 such that the angle $\measuredangle(v_1, 0, v_2)$ is not defined.

Exercise 2.7 (30 points). Let Γ be a group acting on the hyperbolic space $\mathbb{H}$ properly discontinuously by isometries. Prove that $\frac{\mathbb{H}}{\Gamma}$ is CAT(0) if and only if Γ is generated by reflections, or find a countrexample.

3 Polyhedral spaces and quasi-isometries

Exercise 3.1 (20 points). Let $A = \langle a, b \ \mid \ aba^{-1}b^{-1} = b\rangle$ be a group with the only relation $aba^{-1}b^{-1} = b$. Prove that A with word metric is not quasi-isometric to $\mathbb{Z}^n$ for any $n \in \mathbb{Z}^{>0}$.

Exercise 3.2 (10 points). Let $\Gamma_0 \subset \Gamma$ be a finite index subgroup. Prove that Γ_0 is quasi-isometric to Γ.

Exercise 3.3 (10 points). Prove that an infinite-dimensional Hilbert space is not quasi-isometric to the hyperbolic space form $\mathbb{H}^n$.

Exercise 3.4 (20 points). Let Γ be a group which does not have elements of infinite order. Prove that its Cayley graph is not quasi-isometric to $\mathbb{R}$.

Exercise 3.5 (10 points). Let M be a simplicial metric space (possibly with infinitely many simplices) where all k-simplices are isometric. Prove that M is geodesic.

Exercise 3.6 (10 points). Let Γ be a finitely-generated nilpotent group, and d its word metric. Denote by $|B_e(N)|$ the number of elements in an open ball with center in e of radius N. Prove that $|B_e(N)| \leqslant P(N)$ for some polynomial N.

Exercise 3.7 (20 points). Let V be a normed vector space which satisfies CAT(0)-inequalities. Prove that the norm on V is Euclidean.

4 Alexandrov spaces and convexity

Exercise 4.1 (10 points). Prove that a bouquet of CAT(0)-spaces is CAT(0).

Exercise 4.2 (10 points). Recall that $X \coprod_Y Z$ is X glued with Z over Y which is isometrically embedded in both. Let $A \subset X$ be a subset in a CAT(0)-space such that $X \coprod_A X$ is CAT(0). Prove that A is convex.

Exercise 4.3 (20 points). Let V be a normed vector space which satisfies CAT(0)-inequalities. Prove that the norm on V is Euclidean.

Exercise 4.4 (20 points). Let M be a Hadamard space, that is, a geodesic, simply connected CAT(0)-space, and $K \subset M$ a convex subset. Prove that he distance to K is a convex function on $M \backslash K$.

Definition 4.1. A function $f: M \longrightarrow \mathbb{R}$ on a metric space is called **λ-convex** if for any geodesic $\gamma: [0,t] \longrightarrow M$, the function $u \longrightarrow f(\gamma(u)) - \lambda u^2$ is convex.

Exercise 4.5 (20 points). Suppose that $\lambda > 0$. Prove that any continuous λ-convex function on a complete geodesic space reaches a local minimum.

Exercise 4.6 (20 points). Let M be a geodesic space. Denote by $K_4(M)$ the subset in $\mathbb{R}^6$, obtained from all 6-tuples $|ab|, |ac|, |ad|, |bc|, |bd|, |cd|$ for all $\{a,b,c,d\} \subset M$. Suppose that $K_4(M) \subset K_4(\mathbb{R}^2)$. Construct an isometry between M and a convex subset in $\mathbb{R}^2$.

Exercise 4.7 (10 points). Let $M = \{v \in \mathbb{R}^n \ \ |v| \geqslant 1\}$ be a Euclidean space without a unit ball. Prove that M is locally CAT(0).

Exercise 4.8 (10 points). Let M be a locally compact, simply connected CAT(0)-space, and Γ a group acting on M by isometries. Prove that any finite orbit of Γ is contained in a compact, contractible, Γ-invariant subset of M.

5 Gromov hyperbolic spaces

Exercise 5.1 (20 points). Let M be a geodesic space such that any loop in M can be contracted inside its δ-neighbourhood. Prove that M is Rips hyperbolic.

Exercise 5.2 (20 points). Let M be a Rips δ-hyperbolic space, and D a geodesic 2^n-polygon. Prove that each side of D lies in an $n\delta$-neighbourhood of the rest.

Definition 5.1. A subset $Z \subset M$ of a geodesic space is called **quasiconvex** if any minimizing geodesic connecting two points of Z belongs to an ε-neighbourhood of Z,

Exercise 5.3 (10 points). Let M be a Rips δ-hyperbolic space. Prove that any open ball in M is quasiconvex.

Exercise 5.4 (10 points). Let M be a Rips δ-hyperbolic geodesic space. Prove that a geodesic in M is quasiconvex.

Exercise 5.5 (20 points). Let x, y, z be points on a boundary of a ball $B_s(r)$ in a Gromov δ-hyperbolic space, and $|xy| = |yz| = d < \frac{1}{10}r$. Prove that $|xz| \leqslant d + 4\delta$

Exercise 5.6 (10 points). Prove that the free product of hyperbolic groups is hyperbolic.

Exercise 5.7 (10 points). Recall that **ultrametric space** is a metric space which satisfies $d(x,y) \leqslant \max(d(x,z), d(y,z))$. Prove that an ultrametric space is Gromov 0-hyperbolic.

Exercise 5.8 (20 points). Let Γ be a nilpotent group, which does not have a cyclic subgroup of finite index. Prove that Γ is not Gromov hyperbolic.