Metric spaces

lecture 1: Length structures

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Metric spaces

DEFINITION: Let M be a set. **A metric** on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$, satisfying the following conditions.
* **[Non-degeneracy:]** $d(x,y) = 0 \Leftrightarrow x = y$.
* **[Symmetry:]** $d(x,y) = d(y,x)$
* **[Triangle inequality:]** $d(x,y) \leqslant d(x,z) + d(z,y)$.

for any $x, y, z \in M$.

DEFINITION: Let $x \in M$ be a point, and $\varepsilon \in \mathbb{R}^{\geqslant 0}$. The set

$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$$

is called **an open ball** of radius ε with center in x, or **an ε-ball**. **A closed ball** is the set $\{y \in M \mid d(x,y) \leqslant \varepsilon\}$.

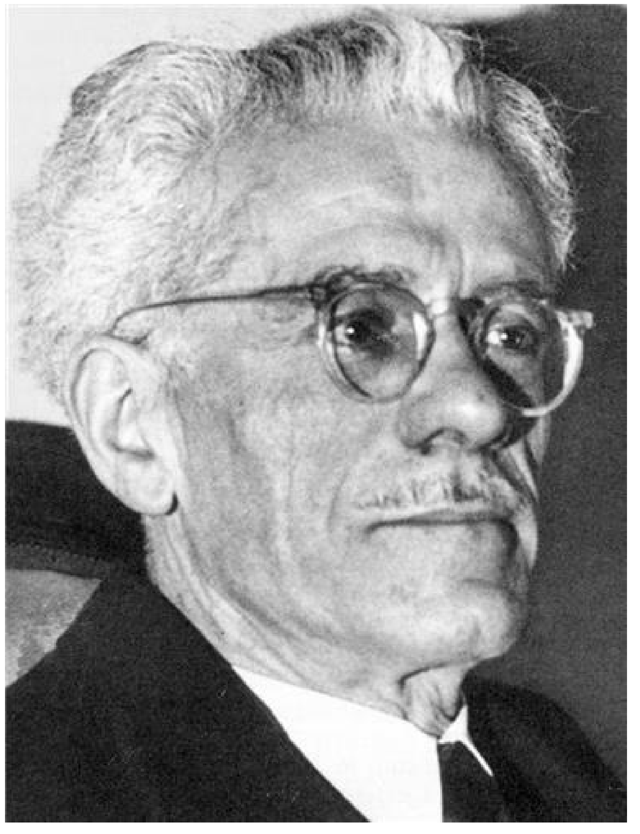
DEFINITION: An open set in a metric space M is a union of open balls.

EXERCISE: Prove that **this defines a topology on M.**

EXERCISE: Prove that **a closed ball is closed** in the sense of this topology.

Maurice Fréchet

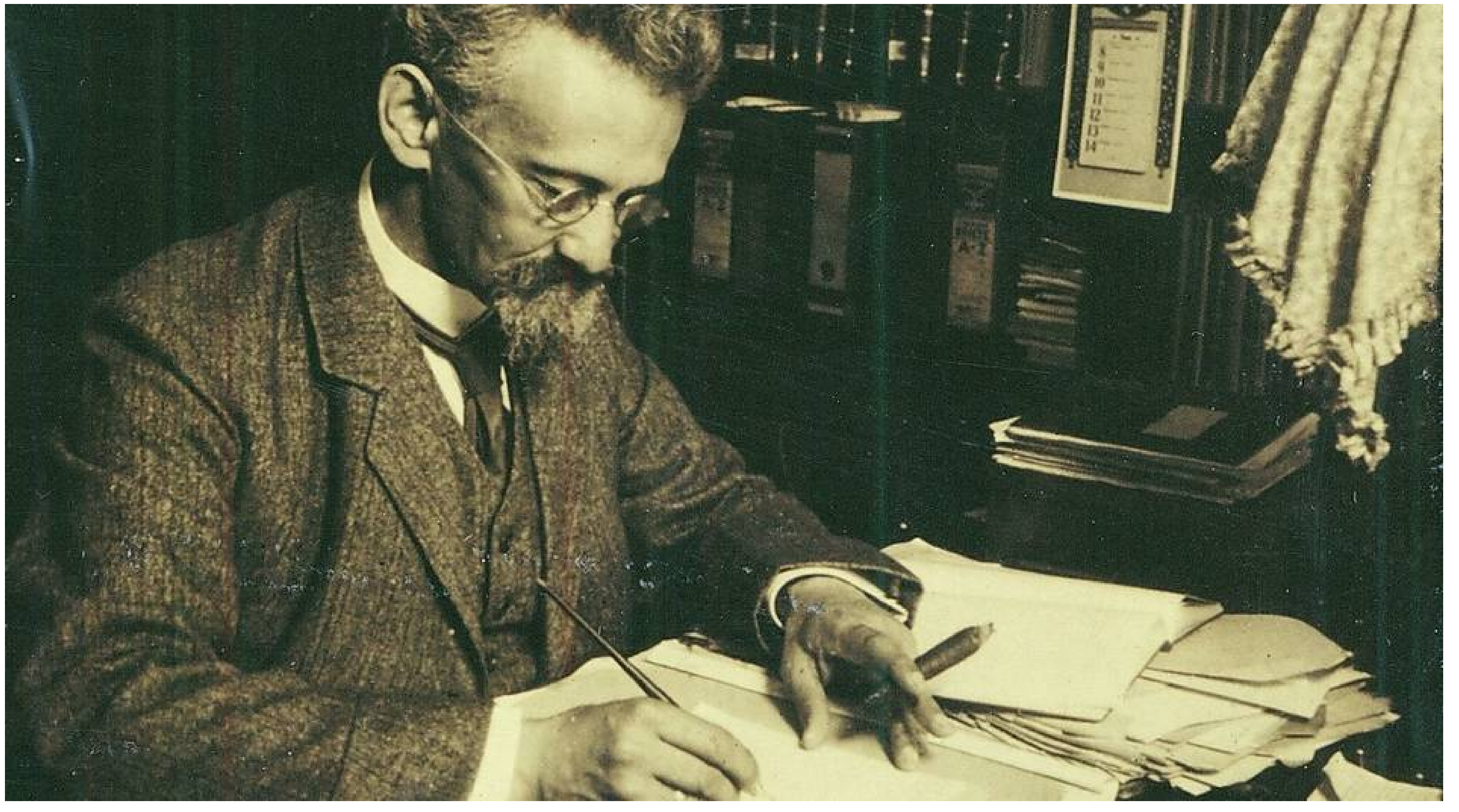
The notion of a metric was defined by Maurice Fréchet, 1905 ("La notion d'ècart et le calcul fonctionnel"), and in his dissertation (1906).



Maurice Fréchet
(1878 – 1973)

Felix Hausdorff

The notions of the metric and the topological space are due to Felix Hausdorff, "Grundzüge der Mengenlehre", 1914.



Felix Hausdorff
(1868 – 1942)

Arc-length of a path

Let (M,d) be a metric space, and $\gamma: [a,b] \mapsto M$ a continuous path (here $[a,b]$ denotes the closed interval). Let $x_0 = a < x_1 < ... < x_{n-1} < b = x_n$ be the partition of the interval, and $L_\gamma(x_1, ...x_{n-1}) := \sum_{i=0}^{n-1} d(\gamma(x_i), \gamma(x_{i+1}))$ the length of the corresponding polygonal chain.

DEFINITION: We define **the arc-length** (or **the length**) of the path γ as

$$L_d(\gamma) := \sup_{a<x_1<...<x_{n-1}<b} L_\gamma(x_1, ...x_{n-1}),$$

where supremum is taken over all partitions of the interval $[a,b]$. A path is called **rectifiable** if its arc-length is finite.

EXERCISE: Prove the following properties of arc-length.
* **The arc-length is additive:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L_d(\gamma) = L_d(\gamma|_{[a,b]}) + L_d(\gamma|_{[b,c]})$.
* **The arc-length is continuous as a function of the ends:** for any rectifiable path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**
* **The arc-length is compatible with the metric:** for any $x,y \in M$, and any path $\gamma: [a,b] \mapsto M$ with ends in x,y, we have $L_d(\gamma) \geqslant d(x,y)$.
* **The arc-length is invariant under the reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ the arc-length of γ is equal to the arc-length of $\varphi \circ \gamma$.

Arc-length of a smooth path

THEOREM: Let $\gamma: [a,b] \longrightarrow \mathbb{R}^n$ be a smooth path in $\mathbb{R}^n$ with the standard metric. **Then** $L_d(\gamma) = \int_a^b |\gamma'(t)|dt$**.**

Proof. Step 1: By Newton-Leibniz, for any $x, y \in [a,b]$, we have

$$\gamma(y) - \gamma(x) = \int_x^y \gamma'(t)dt,$$

hence $d(\gamma(x), \gamma(y)) \leqslant \int_x^y |\gamma'(t)|dt$, which gives an inequality

$$L_\gamma(x_1, ... x_{n-1}) = \sum_{i=0}^{n-1} d(\gamma(x_i), \gamma(x_{i+1})) \leqslant \sum \int_{x_i}^{x_{i+1}} |\gamma'(t)|dt = \int_a^b |\gamma'(t)|dt.$$

Then $L_d(\gamma) \leqslant \int_a^b |\gamma'(t)|dt$.

Arc-length of a smooth path (2)

Step 2: The converse inequality is proven as follows. We partition $[a,b]$ onto intervals $[x_i, x_{i+1}]$. From the Taylor expansion with remainder, we obtain

$$\left|\gamma'(t_0) - \frac{\gamma(x_{i+1}) - \gamma(x_i)}{x_{i+1} - x_i}\right| \leqslant C|x_{i+1} - x_i|,$$

for all $t \in [x_i, x_{i+1}]$ where $C = \sup|\gamma''|$. Then

$$\Big||\gamma'(t)|(x_{i+1} - x_i) - |\gamma(x_i) - \gamma(x_{i+1})|\Big| \leqslant C|x_{i+1} - x_i|^2. \quad (*)$$

The average of $|\gamma'(t)|$ on $[x_i, x_{i+1}]$ is equal to

$$\mathsf{Av}_{t\in[x_i,x_{i+1}]}|\gamma'(t)| = (x_{i+1} - x_i)^{-1}\int_{x_i}^{x_{i+1}} |\gamma'(t)|dt,$$

hence (*) implies

$$\left|\int_{x_i}^{x_{i+1}} |\gamma'(t)|dt - |\gamma(x_i) - \gamma(x_{i+1})|\right| \leqslant C|x_{i+1} - x_i|^2. \quad (**)$$

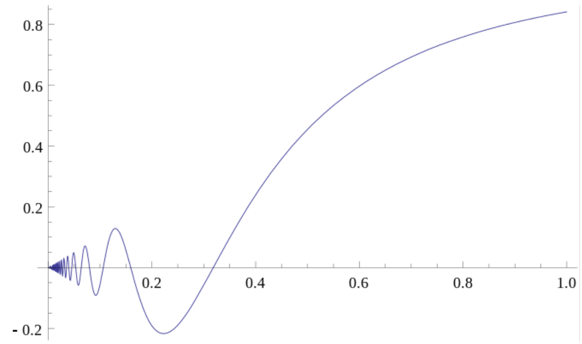
Step 3: Choosing $|x_{i+1} - x_i| < \varepsilon$ and using (**), we obtain that

$$L_\gamma(x_1, ...x_{n-1}) \geqslant \sum_i \int_{x_i}^{x_{i+1}} |\gamma'(t)|dt - \sum_i C|x_{i+1} - x_i|^2 \geqslant \left|\int_{x_i}^{x_{i+1}} |\gamma'(t)|dt\right| - |a-b|C\varepsilon,$$

because $\sum_i C|x_{i+1} - x_i|^2 \leqslant |a-b|C\varepsilon$. Since $L_d(\gamma) = \sup L_\gamma(x_1, ...x_{n-1})$, this implies that $L_d(\gamma) \geqslant \sum_i \int_{x_i}^{x_{i+1}} |\gamma'(t)dt| = \int_a^b |\gamma'(t)|dt$. ■

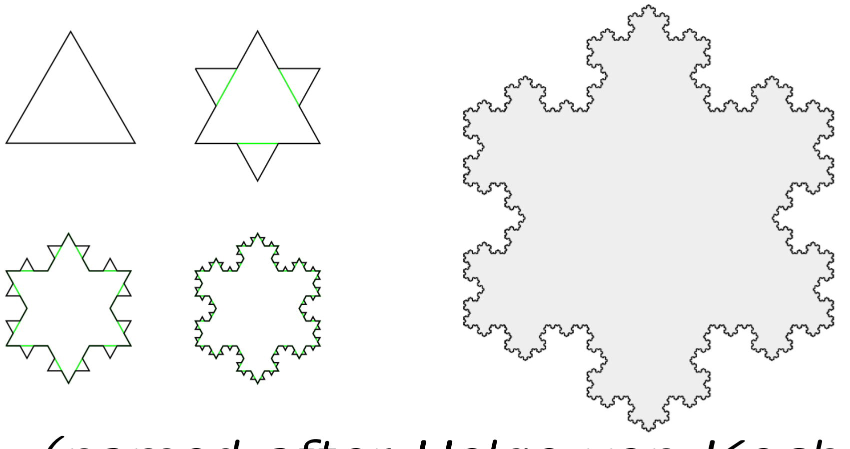
Non-rectifiable curves

EXAMPLE: The curve $\gamma: [0,1] \longrightarrow \mathbb{R}^2$ taking t to $(t, t\sin(t^{-1}))$ **is non-rectifiable**, because $\gamma' = \left(1, \sin(t^{-1}) - \frac{\cos(t^{-1})}{t}\right)$ and $\int_0^1 \left|\frac{\cos(t^{-1})}{t}\right| dt = \infty$ (it is bounded from below by $const \sum_n \frac{1}{n}$).



The graph of $t \mapsto t\sin(t^{-1})$.

EXAMPLE: The Koch curve is non-rectifiable.



The Koch curve *(named after Helge von Koch, 1870-1924).*

Geodesics

DEFINITION: A path $\gamma : [a,b] \longrightarrow M$ is called **a minimizing geodesic** if $d(\gamma(x),\gamma(y)) = |x-y|$ for all $x,y \in [a,b]$. In other words, **a map $\gamma : [a,b] \longrightarrow M$ is a minimizing geodesic if and only if it is an isometry.**

REMARK: **A geodesic** is a map $\gamma : [a,b] \longrightarrow M$ such that $[a,b]$ is a union of intervals $[a,b] = \bigcup_i [x_i, x_{i+1}]$, and $\gamma\Big|_{[x_i,x_{i+1}]}$ is a minimizing geodesic. This definition is very important in differential geometry, but we (almost) never will use it.

REMARK: We are interested in metric spaces where every two points can be connected by a minimizing geodesic; such metrics are called **geodesic metrics**. Riemannian metric on a complete Riemannian manifolds have this property. In the next lecture I will explain how to construct such metrics.

Intrinsic metric

DEFINITION: A metric d on M us called **intrinsic metric**, if $d(x,y) = \inf_\gamma L_d(\gamma)$, where the infimum is taken over all rectifiable paths γ connecting x to y.

REMARK: Geodesic metrics are clearly intrinsic. **Hopf-Rinow theorem** claims that the converse is also true when M is locally compact: **in a locally compact, complete metric space with intrinsic metric, any two points can be connected by a minimizing geodesic.** I will give its proof later in this course.

In his book "Metric structures for Riemannian and non-Riemannian spaces", Gromov realized that **it is easier to define metrics in terms of what he called "length structures", by axiomatizing the properties of arc-length.** This approach is convenient, because it allows to give many examples of intrinsic metric spaces immediately.

Admissible paths

DEFINITION: Let M be a topological space. **A class of admissible paths** is a set $\mathcal{C}$ of continuous maps $[a,b] \longrightarrow M$ with the following properties.

1. **The concatenation.** For any two paths $[a,b] \xrightarrow{\gamma_1} M$, $[b,c] \xrightarrow{\gamma_2} M$, satisfying $\gamma_1(b) = \gamma_2(b)$, **the concatenation** $\gamma: [a,c] \longrightarrow M$ (that is, the path equal to γ_1 on $[a,b]$ and to γ_2 on $[b,c]$) is also admissible.

2. **Linear reparametrization.** For any linear map $\varphi: [a,b] \longrightarrow [c,d]$ and any admissible path $\gamma: [c,d] \longrightarrow M$, the path $\varphi \circ \gamma$ is also admissible.

3. **Restriction.** For each path $[a,b] \xrightarrow{\gamma} M$, and an interval $[c,d] \subset [a,b]$, the restriction $\gamma|_{[c,d]}$ is also admissible.

Admissible paths (examples)

EXAMPLE: Polygonal chains in $\mathbb{R}^n$ **constitute an admissible class.**

EXAMPLE: Piecewise polynomial paths are obtained by concatenation of a collection of paths $\gamma_i : [x_i, x_{i+1}]$, given by polynomial maps $R \mapsto \mathbb{R}^n$. Clearly, **piecewise polynomial paths constitute an admissible class.**

EXAMPLE: Piecewise smooth paths are obtained by concatenation of a collection of paths $\gamma_i : [x_i, x_{i+1}]$, given by smooth maps $R \mapsto \mathbb{R}^n$. Clearly, **piecewise smooth paths constitute an admissible class.**

DEFINITION: A **C-Lipschitz map** is a map of metric spaces $\varphi : M \longrightarrow M'$ satisfying $Cd(x,y) \geqslant d(\varphi(x), \varphi(y))$. **Lipschitz map** is a C-Lipschitz map with some constant C.

EXAMPLE: Lipschitz paths in a metric space constitute an admissible class

EXAMPLE: Rectifiable paths in a metric space constitute an admissible class

Length structures

DEFINITION: Let M be a topological space. **A length structure** on M is a class $\mathcal{C}$ of admissible paths together with **a length functional** $L: \mathcal{C} \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$ which satisfies the following axioms.

1. **Additivity:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L(\gamma) = L(\gamma|_{[a,b]}) + L(\gamma|_{[b,c]})$.

2. **The length is continuous as a function of the ends:** for any path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**

3 **Invariance under reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ and any admissible path $\gamma: [a,c] \longrightarrow M$ such that $\varphi \circ \gamma$ is also admissible, one has $L(\gamma) = L(\varphi \circ \gamma)$.

4. **Compatibility with the topology:** for any point $x \in M$, and any closed set $Z \subset M$, such that $x \notin Z$, there is a number $\varepsilon > 0$ such that $L(\gamma) > \varepsilon$ for any admissible path connecting Z to x.

EXAMPLE: The arc-length on the class of rectifiable paths **gives a length structure.**

The metric associated with a length structure

DEFINITION: Let M be a topological space equipped with a length structure L. The **path metric** d_L associated with L is defined as $d_L(x,y) := \inf_\gamma L(\gamma)$, where the infimum is taken over all admissible paths connecting x tp y.

CLAIM: d_L is a metric.

Proof: **d_L is symmetric** because the parameter change $t \longrightarrow (b-t)+a$ takes a path connecting x to y to a path connecting y to x, and does not change the length. **Positivity of $d_L(x,y)$ for all $x \neq y$** follows from the condition 4, applied when $Z = \{y\}$. Indeed, there exists a number $\varepsilon > 0$ such that any path γ connecting x to $Z = \{y\}$ satisfies $L(\gamma) > 0$, hence $d_L(x,y) > \varepsilon$. **Triangle inequality follows from the concatenation**. Let γ_1 be a path connecting x to y, and γ_1 a path connecting y to z. We can chose γ_1, γ_2 such that $L(\gamma_1) < d_L(x,y) + \varepsilon$, and $L(\gamma_2) < d_L(y,z) + \varepsilon$, for any given $\varepsilon > 0$. Denote by γ the concatenation of γ_1 and γ_2, it is an admissible path connecting x to z, and satisfies $L(\gamma) = L(\gamma_1) + L(\gamma_2) < d_L(x,y) + d_L(y,z) + 2\varepsilon$ This gives

$$d_L(x,y) + d_L(y,z) \geqslant L(\gamma) + 2\varepsilon \geqslant d_L(x,z) + 2\varepsilon.$$

■

In the next lecture **we will prove that d_L is an intrinsic metric.**