Metric spaces

lecture 2: Intrinsic metrics

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Admissible paths

DEFINITION: Let M be a topological space. **A class of admissible paths** is a set $\mathcal{C}$ of continuous maps $[a,b] \longrightarrow M$ with the following properties.

1. **The concatenation.** For any two paths $[a,b] \xrightarrow{\gamma_1} M$, $[b,c] \xrightarrow{\gamma_2} M$, satisfying $\gamma_1(b) = \gamma_2(b)$, **the concatenation** $\gamma: [a,c] \longrightarrow M$ (that is, the path equal to γ_1 on $[a,b]$ and to γ_2 on $[b,c]$) is also admissible.

2. **Linear reparametrization.** For any linear map $\varphi: [a,b] \longrightarrow [c,d]$ and any admissible path $\gamma: [c,d] \longrightarrow M$, the path $\varphi \circ \gamma$ is also admissible.

3. **Restriction.** For each path $[a,b] \xrightarrow{\gamma} M$, and an interval $[c,d] \subset [a,b]$, the restriction $\gamma|_{[c,d]}$ is also admissible.

Admissible paths (examples)

EXAMPLE: Polygonal chains in $\mathbb{R}^n$ **constitute an admissible class.**

EXAMPLE: Piecewise polynomial paths are obtained by concatenation of a collection of paths $\gamma_i : [x_i, x_{i+1}]$, given by polynomial maps $R \mapsto \mathbb{R}^n$. Clearly, **piecewise polynomial paths constitute an admissible class.**

EXAMPLE: Piecewise smooth paths are obtained by concatenation of a collection of paths $\gamma_i : [x_i, x_{i+1}]$, given by smooth maps $R \mapsto \mathbb{R}^n$. Clearly, **piecewise smooth paths constitute an admissible class.**

DEFINITION: A **C-Lipschitz map** is a map of metric spaces $\varphi : M \longrightarrow M'$ satisfying $Cd(x,y) \geqslant d(\varphi(x), \varphi(y))$. **Lipschitz map** is a C-Lipschitz map with some constant C.

EXAMPLE: Lipschitz paths in a metric space constitute an admissible class

EXAMPLE: Rectifiable paths in a metric space constitute an admissible class

Length structures

DEFINITION: Let M be a topological space. **A length structure** on M is a class $\mathcal{C}$ of admissible paths together with **a length functional** $L: \mathcal{C} \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$ which satisfies the following axioms.

1. **Additivity:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L(\gamma) = L(\gamma|_{[a,b]}) + L(\gamma|_{[b,c]})$.

2. **The length is continuous as a function of the ends:** for any path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**

3 **Invariance under reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ and any admissible path $\gamma: [a,c] \longrightarrow M$ such that $\varphi \circ \gamma$ is also admissible, one has $L(\gamma) = L(\varphi \circ \gamma)$.

4. **Compatibility with the topology:** for any point $x \in M$, and any closed set $Z \subset M$, such that $x \notin Z$, there is a number $\varepsilon > 0$ such that $L(\gamma) > \varepsilon$ for any admissible path connecting Z to x.

EXAMPLE: The arc-length on the class of rectifiable paths **gives a length structure.**

The metric associated with a length structure

DEFINITION: Let M be a topological space equipped with a length structure L. The **path metric** d_L associated with L is defined as $d_L(x,y) := \inf_\gamma L(\gamma)$, where the infimum is taken over all admissible paths connecting x tp y.

CLAIM: d_L is a metric.

Proof: **d_L is symmetric** because the parameter change $t \longrightarrow (b-t)+a$ takes a path connecting x to y to a path connecting y to x, and does not change the length. **Positivity of $d_L(x,y)$ for all $x \neq y$** follows from the condition 4, applied when $Z = \{y\}$. Indeed, there exists a number $\varepsilon > 0$ such that any path γ connecting x to $Z = \{y\}$ satisfies $L(\gamma) > 0$, hence $d_L(x,y) > \varepsilon$. **Triangle inequality follows from the concatenation**. Let γ_1 be a path connecting x to y, and γ_1 a path connecting y to z. We can chose γ_1, γ_2 such that $L(\gamma_1) < d_L(x,y) + \varepsilon$, and $L(\gamma_2) < d_L(y,z) + \varepsilon$, for any given $\varepsilon > 0$. Denote by γ the concatenation of γ_1 and γ_2, it is an admissible path connecting x to z, and satisfies $L(\gamma) = L(\gamma_1) + L(\gamma_2) < d_L(x,y) + d_L(y,z) + 2\varepsilon$ This gives

$$d_L(x,y) + d_L(y,z) \geqslant L(\gamma) + 2\varepsilon \geqslant d_L(x,z) + 2\varepsilon.$$

■

Examples of length structures

EXAMPLE: Let $M = \mathbb{R}^n$ with the standard topology, and $\mathcal{C}$ the class of all piecewise-linear paths (polygonal chains) $[x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$. Define the length functional taking $\gamma = [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n] \in \mathcal{C}$ to $L(\gamma) = \sum |d(x_i, x_{i+1})|$.

CLAIM: The metric d_L constructed from this length structure **is equal to the standard Euclidean metric.**

Proof: Indeed, **the shortest polygonal chain connecting two points is an interval of a line.** ■

EXAMPLE: ("crossing of a swamp": conformally flat metric)
Let $M = \mathbb{R}^n$ and $\mathcal{C}$ the class of all piecewise-linear paths (polygonal chains). Fix a continuous, positive function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^{>0}$. Define the length functional as

$$L(\gamma) = \sum \int_{[x_i, x_{i+1}]} f dt$$

where dt is the unit 1-form on the interval.

EXERCISE: Prove that this defines a length structure.

Finsler metrics

EXAMPLE: Let M be $\mathbb{R}^n$, and $\mathcal{C}$ the class of all piecewise smooth paths. Define the length functional as $L(\gamma(t)) := \int_a^b |\gamma'(t)| dt$.

CLAIM: This functional defines a length structure.
Proof: Indeed, it is equal to the arc length. ■

EXAMPLE: (**A Finsler metric**)
Let $U \subset \mathbb{R}^n$ be an open set, and $\nu_x : T_x\mathbb{R}^n \longrightarrow \mathbb{R}$ the norm on the tangent space, continuously depending on x. For any piecewisely smooth path $\gamma : [a, b] \longrightarrow U$, define $L_\nu(\gamma(t)) := \int_a^b \nu_{\gamma(t)}(\gamma'(t)) dt$

PROPOSITION: This defines a length functional on the class of piecewisely smooth paths.

Proof: Additivity and continuity of L are clear. Compatibility with topology is implied by $L_\nu \geqslant L_d$, where L_d is the arc length associated with an Euclidean metric such that the corresponding norm on $T_x\mathbb{R}^n$ is smaller than ν_x.

Finsler metrics and Riemannian metrics

Invariance of L_ν under the reparametrization is implied by the formula

$$L_\nu(\varphi\circ\gamma) = \int_a^b \nu_{\gamma(\varphi(t))}(\varphi\circ\gamma)'(t)dt =$$
$$= \int_a^b \nu_{\gamma(\varphi(t))}(\varphi'(t)\gamma'(\varphi(t))dt = \int_a^b \varphi'(t)\nu_{\gamma(\varphi(t))}(\gamma'(\varphi(t))dt =$$
$$= \int_{\varphi(a)}^{\varphi(b)} \nu_{\gamma(\varphi(t))}(\gamma'(\varphi(t))d\varphi(t) = L_\nu(\gamma)$$

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DEFINITION: Let M be a manifold, and $\nu_x: T_x\mathbb{R}^n \longrightarrow \mathbb{R}$ the norm on the tangent space, continuously depending on x. Define a functional L_ν on the class of piecewise smooth paths by

$$L_\nu(\gamma(t)) := \int_a^b \nu_{\gamma(t)}(\gamma'(t))dt.$$

The corresponding path metric on M is called **a Finsler metric**. When the norm ν_x is Euclidean for all points, this metric is called **a Riemannian metric**, and the Euclidean scalar product g_x **a Riemannian form.** Usually it is considered as a section of the bundle $\operatorname{Sym}^2(T^*M)$ of symmetric 2-forms on M.

Geodesics (reminder)

DEFINITION: A path $\gamma: [a,b] \longrightarrow M$ is called **a minimizing geodesic** if $d(\gamma(x),\gamma(y)) = |x-y|$ for all $x, y \in [a,b]$. In other words, **a map $\gamma: [a,b] \longrightarrow M$ is a minimizing geodesic if and only if it is an isometry.**

REMARK: **A geodesic** is a map $\gamma: [a,b] \longrightarrow M$ such that $[a,b]$ is a union of intervals $[a,b] = \bigcup [x_i, x_{i+1}]$, and $\gamma\Big|_{[x_i,x_{i+1}]}$ is a minimizing geodesic. This definition is very important in differential geometry, but we (almost) never will use it.

REMARK: We are interested in metric spaces where every two points can be connected by a minimizing geodesic; such metrics are called **geodesic metrics**. Riemannian metric on a complete Riemannian manifolds have this property. In the next lecture I will explain how to construct such metrics.

Arc-length and rectifiable paths (reminder)

Let (M,d) be a metric space, and $\gamma: [a,b] \mapsto M$ a continuous path (here $[a,b]$ denotes the closed interval). Let $x_0 = a < x_1 < ... < x_{n-1} < b = x_n$ be the partition of the interval, and $L_\gamma(x_1, ... x_{n-1}) := \sum_{i=0}^{n-1} d(\gamma(x_i), \gamma(x_{i+1}))$ the length of the corresponding polygonal chain.

DEFINITION: We define **the arc-length** (or **the length**) of the path γ as

$$L_d(\gamma) := \sup_{a<x_1<...<x_{n-1}<b} L_\gamma(x_1, ... x_{n-1}),$$

where supremum is taken over all partitions of the interval $[a,b]$. A path is called **rectifiable** if its arc-length is finite.

CLAIM: Let $\mathcal{C}$ be the class of rectifiable paths, and $L_d: \mathcal{C} \longrightarrow \mathbb{R}$ the arc-length. Then $(\mathcal{C}, L_d)$ is a length structure.

Arc-length as a length structure

DEFINITION: Let (M,d) be a metric space, and $L_d: \mathcal{C} \longrightarrow \mathbb{R}$ the arc-length functional on the class of rectifiable paths. Denote by $\hat{d}$ the corresponding path metric. It is called **the path metric, induced by d.**

REMARK: Clearly, $\hat{d} \geqslant d$. Indeed, **for any path γ connecting x to y, $L_d(\gamma) \geqslant d(x,y)$ by the triangle inequality.**

DEFINITION: A metric d on M is called **intrinsic** of $\hat{d} = d$.

THEOREM: For any metric space (M,d), **the metric $\hat{d}$ is intrinsic,** that is, $\hat{\hat{d}} = \hat{d}$.

Proof. Step 1: Since $\hat{d} \geqslant d$, we have $L_d(\gamma) \leqslant L_{\hat{d}}(\gamma)$ for any path γ in M.

Step 2: Let $\gamma: [a,b] \to M$ be a rectifiable path. Take a partition $x_1, ..., x_{n-1}$ of the interval such that $L_{\hat{d}}(\gamma) - \sum \hat{d}(\gamma(x_i), \gamma(x_{i+1})) < \varepsilon$. Then

$$L_{\hat{d}}(\gamma) - \varepsilon \leqslant \sum_i \hat{d}\left(\gamma\Big|_{[x_i,x_{i+1}]}\right) \leqslant \sum_i L_d\left(\gamma\Big|_{[x_i,x_{i+1}]}\right) = L_d(\gamma).$$

Passing to the limit $\varepsilon \to 0$, we obtain $L_{\hat{d}}(\gamma) \leqslant L_d(\gamma)$. Since $\hat{d} \geqslant d$, his gives $L_{\hat{d}}(\gamma) = L_d(\gamma)$, hence $\hat{\hat{d}} = \hat{d}$. ■

Intrinsic metrics and length structures

THEOREM: Let $(M, \mathcal{C}, L)$ be a topological space equipped with a length structure, and d the associated path metric. **Then d is intrinsic.**

Proof. Step 1: Let γ be an admissible path connecting a and b in M. By definition, $d(a,b) \leqslant L(\gamma)$. On the other hand, $L_d(\gamma)$ is the supremum of $\sum_i d(\gamma(x_i), \gamma(x_{i+1}))$ for all partitions of $[a,b]$. Therefore, for each $\varepsilon > 0$ there exists a partition $x_1, ..., x_{n-1}$ of $[a,b]$ such that

$$L_d(\gamma) \leqslant \sum_i d(\gamma(x_i), \gamma(x_{i+1})) + \varepsilon \leqslant \sum_i L(\gamma_{[x_i, x_{i+1}]}) + \varepsilon = L(\gamma) + \varepsilon.$$

Passing to the limit $\varepsilon \to 0$, we obtain $L_d(\gamma) \leqslant L(\gamma)$, hence $\hat{d} \leqslant d$.

Step 2: The converse inequality $\hat{d} \geqslant d$ is clear. ■

The following version of Hopf-Rinow theorem is due to Stefan Cohn-Vossen.

THEOREM: Let (M, d) be a complete, locally compact space with intrinsic metric. Then **any two points of M can be connected by a minimizing geodesic.** Moreover, **any closed ball in M is compact.**

Proof: Next lecture. ■