Metric spaces

lecture 3: Local metrics

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Intrinsic metrics (reminder)

Let (M, d) be a metric space, and $\gamma : [a, b] \mapsto M$ a continuous path (here $[a, b]$ denotes the closed interval). Let $x_0 = a < x_1 < ... < x_{n-1} < b = x_n$ be the partition of the interval, and $L_\gamma(x_1, ...x_{n-1}) := \sum_{i=0}^{n-1} d(\gamma(x_i), \gamma(x_{i+1}))$ the length of the corresponding polygonal chain.

DEFINITION: We define **the arc-length** (or **the length**) of the path γ as

$$L_d(\gamma) := \sup_{a<x_1<...<x_{n-1}<b} L_\gamma(x_1, ...x_{n-1}),$$

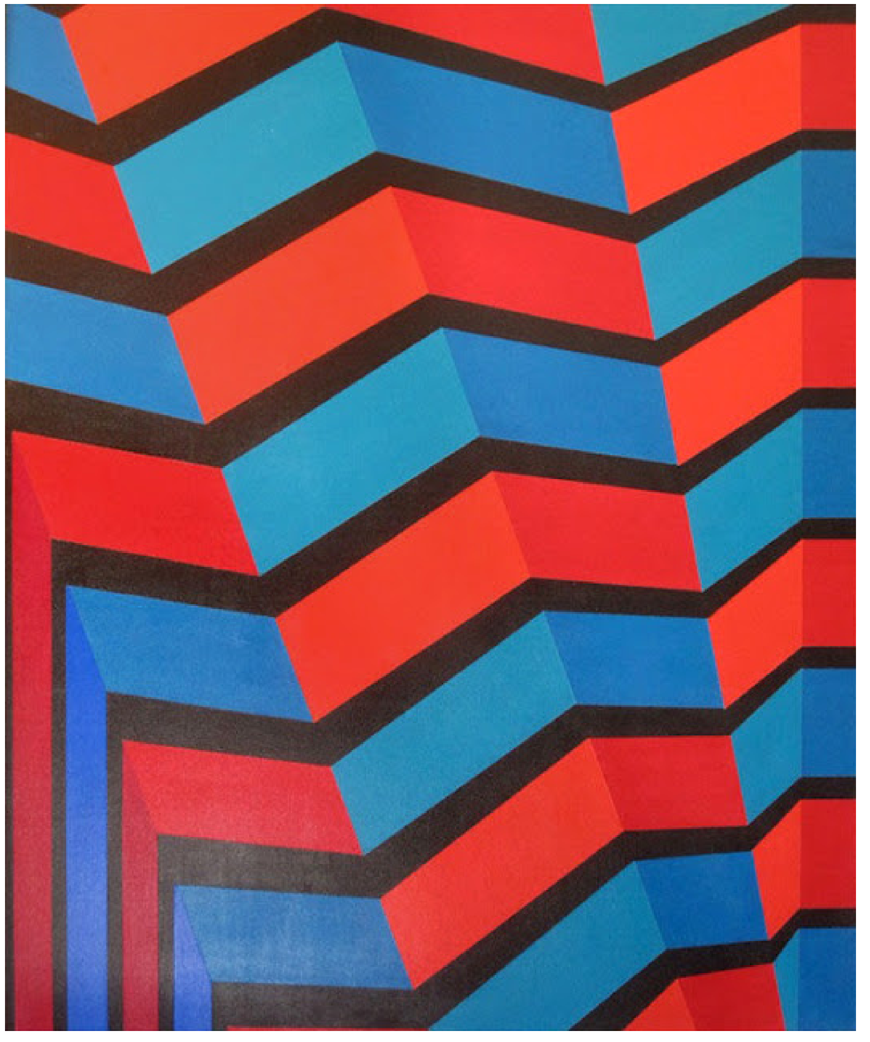
where supremum is taken over all partitions of the interval $[a, b]$. A path is called **rectifiable** if its arc-length is finite.

DEFINITION: A metric d on M us called **intrinsic metric** if $d(x, y) = \inf_\gamma L_d(\gamma)$, where the infimum is taken over all rectifiable paths γ connecting x to y.

Herbert Busemann

Herbert Busemann (1905 – 1994)

Busemann defined the intrinsic metric, Finsler manifolds, and many other concepts of metric geometry.



Herbert Busemann: "Untitled" (1972),
Noho Modern gallery, Los Angeles

Conflict

Herbert Busemann: "Conflict" (1972),
Noho Modern gallery, Los Angeles

Weakly intrinsic metrics

DEFINITION: For any two points x, y in a metric space (M, d), **an ε-chain**, connecting x to y is a collection of points $x = z_0, z_1, ..., z_{n-1}, z_n = y$ such that $d(z_i, z_{i+1}) \leqslant \varepsilon$. Its **defect** is the number $\sum_{i=0}^{n-1} d(z_i, z_{i+1}) - d(x, y)$. The space (M, d) is **weakly intrinsic** if for any two points $x, y \in M$ such that $d(x, y) < \infty$ and any $\varepsilon > 0$, $\delta > 0$, there exists an ε-chain connecting x to y with defect $\leqslant \delta$.

CLAIM: Intrinsic metrics are weakly intrinsic.

Proof. Step 1: For any partition of a recrifiable path $\gamma : [a, b] \longrightarrow M$ **there exists a sub-partition $t_0 = a < t_1 < t_2 < ... < t_n = b$ such that $d(\gamma(t_i), \gamma(t_{i+1})) < \varepsilon$.** To find such a smaller partition, we cover the image of γ by ε-balls and find a finite subcover.

Step 2: Choose a path γ connecting x to y such that $d(x, y) < L_d(\gamma) - \frac{1}{2}\delta$. Take a partition $x_1, ..., x_n$ of γ such that $L_\gamma(x_1, ..., x_m) + \frac{1}{2}\delta > L_d(\gamma)$; the same is true for any sub-partition, $t_1, ..., t_n$, because $L_\gamma(x_1, ..., x_m) \leqslant L_\gamma(t_1, ..., t_n) \leqslant L_d(\gamma)$. **Then $|L_\gamma(t_1, ..., t_n) - L_d(\gamma)| < \frac{1}{2}\delta$ and $|d(x, y) - L_d(\gamma)| < \frac{1}{2}\delta$, hence $|L_\gamma(t_1, ..., t_n) - d(x, y)| < \delta$.**

Weakly intrinsic metrics (2)

Step 3: Using Step 1, we choose a sub-partition $u_1, ..., u_n$ of the partition constructed in Step 2 in such a way that $d(\gamma(u_i), \gamma(u_{i+1})) < \varepsilon$. After passing to a sub-partition, the number $\sum_i d(\gamma(u_i), \gamma(u_{i+1}))$ would possibly increase, hence the property

$$d(x,y) + \delta \geqslant L_d(\gamma) \geqslant \sum_i d(\gamma(u_i), \gamma(u_{i+1})) > d(x,y) - \delta$$

is retained. Then $\gamma(u_0), \gamma(u_1), ...$ is an ε-chain with defect at most δ. ■

Local metrics

CLAIM: Let d_i be a family of metrics (possibly infinite), and $d(x,y) := \sup_i d_i(x,y)$. **Then d is also a metric.**

Proof: We need to check only that $d(x,y) \leqslant d(x,z) + d(z,y)$. This is clear, because

$$d(x,y) = \sup_i d_i(x,y) \leqslant \sup_i (d_i(x,z) + d_i(z,y)) \leqslant$$
$$\leqslant \sup_i d_i(x,z) + \sup_i d_i(z,y) = d(x,z) + d(z,y).$$

■

DEFINITION: Let $\{U_i\}$ be an open covering of s metric space $\{M,d\}$. Denote by $d_{\{U_i\}}$ the metric $\sup_\alpha d_\alpha$, where the supremum is taken over all metrics d_α which satisfy $d_\alpha\big|_{U_i} = d$ for all open sets U_i in the cover. A metric d is called **$\{U_i\}$-local** if $d_{\{U_i\}} = d$. It is called **ε-local**, if it is $\{U_i\}$-local with respect to the covering $\{U_i\}$ consisting of all ε-balls, and **local** if it is ε-local for all $\varepsilon > 0$.

REMARK: This definition is useful if we have a covering $\pi: \tilde M \longrightarrow M$, and **want to extend a metric from M to $\tilde M$.**

EXERCISE: Find all finite local metrics on a line $\mathbb{R}$. Construct a non-local metric on $\mathbb{R}$.

Weakly intrinsic implies local

THEOREM: Let (M,d) be a metric space, with d weakly intrinsic. **Then (M,d) is local.**

Proof. Step 1: Denote by d_ε the supremum of all metrics which are equal to d on all ε-balls. Clearly, d is local if and only if $d=d_\varepsilon$ for all $\varepsilon>0$.

Step 2: Let $x,y\in M$, $d(x,y)<\infty$, and $x=z_0,z_1,...,z_{n-1},z_n=y$ be an ε-chain with defect $\leqslant\delta$. Then

$$d(x,y)\leqslant d_\varepsilon(x,y)\leqslant \sum_{i=0}^{n-1} d_\varepsilon(z_i,z_{i+1}) = \sum_{i=0}^{n-1} d(z_i,z_{i+1})\leqslant d(x,y)+\delta$$

(the equality is true because $d_\varepsilon=d$ on any ε-ball). Passing to the limit $\delta\to 0$, we obtain $d(x,y)=d_\varepsilon(x,y)$. ■

Local implies weakly intrinsic

The converse is also true.

THEOREM: Let (M,d) be a metric space, with d local. **Then (M,d) is weakly intrinsic.**

Proof: Define $d'_\varepsilon(x,y)$ as infimum of $\sum_{i=0}^{n-1} d(z_i,z_{i+1})$ for all ε-chains $z_0 = x, z_1, ..., z_n = y$. Clearly, $d = d'_\varepsilon$ on all $\frac{1}{2}\varepsilon$-balls; since d is local, this implies $d'_\varepsilon \leqslant d$. On the other hand $d'_\varepsilon \geqslant d$ by triangle inequality. This gives $d = d'_\varepsilon$. Then $d(x,y) = \inf_{x_1,...,x_{n-1}} \sum_{i=0}^{n-1} d(z_i,z_{i+1})$ where the infimum is taken over all sequences $z_0 = x, z_1, ..., z_n = y$ such that $d(z_i,z_{i+1}) < \varepsilon$, hence d is weakly intrinsic. ■

The distance between metric balls

DEFINITION: For any two subsets $A, B \subset M$, we denote by $d(A, B)$ the number $\inf_{a\in A, b\in B} d(a, b)$.

DEFINITION: We say that a metric space (M, d) **admits ε-midpoints** if any $x, y \in M$ we have $d(B_x(r/2), B_y(r/2)) = 0$.

THEOREM: Let (M, d) be a metric space. **Then the following conditions are equivalent.**
(1). (M, d) is weakly intrinsic.
(2). (M, d) is local.
(3). For any $x, y \in M$, and any $r_1, r_2 > 0$ such that $d(x, y) = r_1 + r_2$, we have $d(B_x(r_1), B_y(r_2)) = 0$.
(4). (M, d) admits ε-midpoints.

REMARK: The first two are equivalent as we have already shown. Also, (3) clearly implies (4). We are going to prove (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1).

The distance between metric balls (2)

Proof. Step 1: Weak intrinsic implies that for all $\varepsilon, \delta > 0$ there exists an ε-chain $x = t_0, ..., t_n = y$ with defect $\leqslant \delta$. Clearly, **the defect is monotonous if we pass from $t_0, ..., t_n$ to a subset with the same ordering.**

Step 2: Let t_k be the last of t_i which belongs to $B_x(r_1)$, and t_l the first of t_i which belongs to $B_y(r_2)$. Since $t_{k+1} \notin B_x(r_1)$, we have $d(x, t_{k+1}) > r_1$, and $d(t_k, t_{k+1}) < \varepsilon$ **implies that $r_1 > d(x, t_k) \geqslant r_1 - \varepsilon$. Similarly, $r_2 > d(y, t_l) \geqslant r_2 - \varepsilon$.**

Step 3: Since the defect of the chain t_0, t_k, t_l, t_n is bounded by δ (Step 1), we have $d(x, t_k) + d(y, t_l) + d(t_l, t_k) < r_1 + r_2 + \delta$. Then Step 2 implies

$$d(t_l, t_k) < r_1 + r_2 + \delta - d(x, t_k) - d(y, t_l) < \delta + 2\varepsilon.$$

We have obtained $d(B_x(r_1), B_y(r_2)) < \delta + 2\varepsilon$. Passing to the limit as $\varepsilon, \delta \to 0$, we get $d(B_x(r_1), B_y(r_2)) = 0$, hence (1) $\Rightarrow$ (3).

The distance between metric balls (3)

Step 4: We are going to prove that (4) $\Rightarrow$ (1) **(existence of ε-midpoints implies that d is weak intrinsic).** Let $x, y \in M$ be a points which satisfy $d(x,y) = r < \infty$. We fix $\varepsilon \gg \delta > 0$. Choose points $x' \in B_x(r/2)$ and $y' \in B_y(r/2)$ such that $d(x', y') < \frac{1}{3}\delta$. Clearly, the defect δ_0 of the chain x, x', y', y is bounded by $\frac{1}{3}\delta$.

Step 5: To prove that (M,d) is weakly intrinsic, **it remains to construct two ε-chains $a_0 = x, a_1, ..., a_l = x'$ and $b_0 = y, b_1, ..., b_m = y'$, such that $d(a_i, a_{i+1}) < \varepsilon$ and $d(b_j, b_{j+1}) < \varepsilon$, and the defect δ_a, δ_b of these chains is bounded by $\frac{1}{3}\delta$.** Then the defect of the chain $x = a_0, ..., a_n = x', y' = b_m, ..., y = b_0$ is bounded by $\sum_i (a_i, a_{i+1}) + \sum_j d(b_j, b_{j+1}) + d(x', y') - r \leqslant \delta_a + d(x, x') + \delta_b + d(y, y') + d(x', y') - r \leqslant \delta_a + \delta_b + \delta_0 \leqslant \delta$.

Step 6: Let $\lceil \alpha \rceil$ denote the smallest integer $u \geqslant \alpha$. We use induction in $m = \lceil \frac{d(x,y)}{\varepsilon} \rceil$. **Assume that an ε-chain with defect $< \delta$ connecting x to y exists whenever $\lceil \frac{d(x,y)}{\varepsilon} \rceil < n$, for any given $\delta > 0$.** Clearly, $\lceil \frac{d(x,x')}{\varepsilon} \rceil < \lceil \frac{d(x,y)}{\varepsilon} \rceil$ unless $\frac{d(x,x')}{\varepsilon} < 1$. In the latter case, we take the ε-chain $x = z_0, x' = z_1, y' = z_2, y = z_3$, and observe that $d(z_0, z_1) + d(z_1, z_2) + d(z_2, z_2) < d(x, y) + \frac{1}{3}\delta$. Otherwise, we apply the induction assumption to find the chains $a_0 = x, a_1, ..., a_l = x'$ and $b_0 = y, b_1, ..., b_m = y'$ (Step 5). ■