# Metric spaces

**lecture 5: Quotient spaces**

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## Local metrics and weakly intrinsic metrics (reminder)

**CLAIM:** Let $d_i$ be a family of metrics (possibly infinite), and $d(x,y) := \sup_i d_i(x,y)$. **Then $d$ is also a metric.**

**Proof:** We need to check only that $d(x,y) \leqslant d(x,z) + d(z,y)$. This is clear, because

$$d(x,y) = \sup_i d_i(x,y) \leqslant \sup_i (d_i(x,z) + d_i(z,y)) \leqslant$$
$$\leqslant \sup_i d_i(x,z) + \sup_i d_i(z,y) = d(x,z) + d(z,y).$$

■

**DEFINITION:** Let $\{U_i\}$ be an open covering of s metric space $\{M,d\}$. Denote by $d_{\{U_i\}}$ the metric $\sup_\alpha d_\alpha$, where the supremum is taken over all metrics $d_\alpha$ which satisfy $d_\alpha\Big|_{U_i} = d$ for all open sets $U_i$ in the cover. A metric $d$ is called **$\{U_i\}$-local** if $d_{\{U_i\}} = d$. It is called **$\varepsilon$-local**, if it is $\{U_i\}$-local with respect to the covering $\{U_i\}$ consisting of all $\varepsilon$-balls, and **local** if it is $\varepsilon$-local for all $\varepsilon > 0$.

**DEFINITION:** For any two points $x, y$ in a metric space $(M,d)$, **an $\varepsilon$-chain**, connecting $x$ to $y$ is a collection of points $x = z_0, z_1, ..., z_{n-1}, z_n = y$ such that $d(z_i, z_{i+1}) \leqslant \varepsilon$. Its **defect** is the number $\sum_{i=0}^{n-1} d(z_i, z_{i+1}) - d(x,y)$. The space $(M,d)$ is **weakly intrinsic** if for any two points $x, y \in M$ such that $d(x,y) < \infty$ and any $\varepsilon > 0$, $\delta > 0$, there exists an $\varepsilon$-chain connecting $x$ to $y$ with defect $\leqslant \delta$.

## Hopf-Rinow theorem (reminder)

**DEFINITION:** For any two subsets $A, B \subset M$, we denote by $d(A,B)$ the number $\inf_{a\in A, b\in B} d(a,b)$.

**DEFINITION:** We say that a metric space $(M,d)$ **admits $\varepsilon$-midpoints** if any $x,y \in M$ we have $d(B_x(r/2), B_y(r/2)) = 0$.

**THEOREM:** Let $(M,d)$ be a metric space. **Then the following conditions are equivalent.**
(1). $(M,d)$ is weakly intrinsic.
(2). $(M,d)$ is local.
(3). For any $x,y \in M$, and any $r_1, r_2 > 0$ such that $d(x,y) = r_1 + r_2$, we have $d(B_x(r_1), B_y(r_2)) = 0$.
(4). $(M,d)$ admits $\varepsilon$-midpoints.

**Proof:** Lecture 3. ■

**THEOREM: (Hopf-Rinow)**
Let $M$ be a complete, locally compact space with a weakly intrinsic metric. **Then every closed metric ball $B_x^{cl}(r)$ in $M$ is compact.**

**Proof:** Lecture 4. ■

## Normal parametrization (reminder)

**DEFINITION:** A continuous path $\gamma : [a, b] \longrightarrow M$ is called **minimizing**, if its arc-length is equal to $d(\gamma(a), \gamma(b))$.

**DEFINITION:** Given a homeomorphism $\varphi : [a, b] \longrightarrow [a, b]$, and a path $\gamma : [a, b] \longrightarrow M$, the composition $\varphi \circ \gamma$ is also a path from $x := \gamma(a)$ to $y := \gamma(b)$. Such a path is called **a reparametrization of $\gamma$**.

**DEFINITION:** Let $\gamma : [a, b] \longrightarrow M$ be a path, and $\gamma_1$ takes $t \in \mathbb{R}^{\geqslant 0}$ to $\gamma(t)$, where $t$ is the minimal of all $x \in [a, b]$ such that $L_d(\gamma\big|_{[a,x]}) = t$ (here, as everywhere, $L_d$ denotes the arc length). When $t_i \in \mathbb{R}^{\geqslant 0}$ is the sequence converging to $t$, the sequence $\gamma(t_i)$ converges to $t$, because $L_d(\gamma_1\big|_{[t_i,t]}) \geqslant d(\gamma_1(t), \gamma_1(t_i))$. Therefore, $\gamma_1$ is continuous. Such a path $\gamma_1$ is called **the normal parametrization** of $\gamma$.

## Normal parametrization and minimizing geodesics (reminder)

**CLAIM:** Let $\gamma : [a,b] \longrightarrow M$ be minimizing, and $\gamma_1 : [0,\alpha] \longrightarrow M$ its normal parametrization. **Then $\gamma_1$ is an isometry.**

**Proof. Step 1:** Any interval of a minimizing path is minimizing. Indeed, consider a partition $[a,c] \cup [b,c] = [a,b]$. Then

$$d(\gamma(a),\gamma(b)) \leqslant L_\gamma(c) \leqslant L_d(\gamma) = d(\gamma(a),\gamma(b)).$$

This gives $L_\gamma(c) = d(\gamma(a),\gamma(c)) + d(\gamma(c),\gamma(b)) = d(\gamma(a),\gamma(b))$. Then

$$L_d(\gamma) = L_d(\gamma\big|_{[a,c]}) + L_d(\gamma\big|_{[c,b]}) \geqslant d(\gamma(a),\gamma(c)) + d(\gamma(c),\gamma(b)) = d(\gamma(a),\gamma(b))$$

hence **the inequalities $L_d(\gamma\big|_{[a,c]}) \geqslant d(\gamma(a),\gamma(c))$ and $L_d(\gamma\big|_{[c,b]}) \geqslant d(\gamma(c),\gamma(b))$ are equalities.**

**Step 2:** Let $v \in [0,\alpha]$. The same argument applied to a partition $[0,v] = [0,u] \cup [u,v]$ would imply that

$$L_d(\gamma_1\big|_{[0,u]}) + L_d(\gamma_1\big|_{[u,v]}) = u + L_d(\gamma_1\big|_{[u,v]}) = L_d(\gamma_1\big|_{[0,v]}) = v,$$

giving $L_d(\gamma_1\big|_{[u,v]}) = v - u$. Then Step 1 gives $v - u = L_d(\gamma_1\big|_{[u,v]}) = d(\gamma_1(u),\gamma_1(v))$, hence $\gamma_1$ is an isometry. ■

## Existence of minimizing geodesics

**DEFINITION:** A **minimizing geodesic** is an isometry $\gamma : [a, b] \longrightarrow M$.

**REMARK:** It is the same as a minimizing path which is naturally parametrized.

**THEOREM:** Let $M$ be a locally compact, complete space with an almost intrinsic metric, and $x_0, x_1 \in M$. **Then there exists a minimizing geodesic, connecting $x_0$ to $x_1$.**

**Proof. Step 1:** Let $d(x_0, x_1) = \alpha$. Rescaling the metric by a factor of $\alpha^{-1}$, we may assume that $\alpha = 1$. Since $d(\overline{B}_{x_0}(1/2), \overline{B}_{x_1}(1/2) = 0$, and both balls are compact, **their intersection is non-empty.** Indeed, the function $y \stackrel{f}{\mapsto} d(y, \overline{B}_{x_1}(1/2))$ is Lipschitz, hence continuous, hence it has a minimum in somewhere on $B_{x_0}(1/2)$; since $d(\overline{B}_{x_0}(1/2), \overline{B}_{x_1}(1/2) = 0$, infimum of $f$ on $\overline{B}_{x_0}(1/2)$ is zero. Choose a point $x_{1/2}$ in $\overline{B}_{x_0}(1/2) \cap \overline{B}_{x_1}(1/2)$. Then the defect of the sequence $x_o, x_{1/2}, x_1$ is equal 0.

## Existence of minimizing geodesics (2)

**THEOREM:** Let $M$ be a locally compact, complete space with an almost intrinsic metric, and $x_0, x_1 \in M$. **Then there exists a minimizing geodesic, connecting $x_0$ to $x_1$.**

**Proof. Step 1:** We assume that $d(x_0, x_1) = 1$. **Then there exists a point $x_{1/2} \in M$ such that $d(x_o, x_{1/2}) = d(x_{1/2}, x_1) = 1/2$.**

**Step 2:** We apply the same argument to the pairs $x_o, x_{1/2}$ and $x_{1/2}, x_1$ to obtain points $x_{1/4}$ and $x_{3/4}$ such that $d(x_o, x_{1/4}) = 1/4$, $d(x_{1/4}, x_{1/2}) = 1/4$, $d(x_{1/2}, x_{3/4}) = 1/4$, $d(x_{3/4}, x_1) = 1/4$. The defect of the sequence $x_o, x_{1/4}, x_{1/2}, x_{3/4}, x_1$ is again equal 0.

Using induction, **we extend this construction to any dyadic rational number $u = \frac{m}{2^n} \in [0,1]$,** and obtain a collection of points $x_u$ such that $d(x_u, x_v) = |u - v|$.

**Step 3:** We obtained an isometric embedding $\varphi_0$ from the set $D_{[0,1]}$ of dyadic rational numbers in $[0,1]$ to $M$. Any isometry can be extended to a metric completion; this gives an isometry $\varphi : [0,1] \mapsto M$. **By definition, $\varphi$ is a minimizing geodesic.** ■

## The quotient topology

**DEFINITION:** Let $M$ be a topological space, and $\sim$ an equivalence relation. A subset $U \subset M/\!\sim$ is **open in the quotient topology** if its preimage in $M$ is open. This defines **the quotient topology** on **the quotient space** $M/\!\sim$.

**A caution:** The quotient space might be non-Hausdorff, even if $M$ is Hausdorff. **Give an example when this happens.**

**DEFINITION:** Let $G$ be a group acting on a topological space $M$. **The quotient space** $M/G$ is the space $M/\!\sim$ of equivalence classes by the relation $x \sim y \Leftrightarrow x \in G \cdot y$. The quotient space is also called **the space of orbits of the action of** $G$.

**REMARK:** Let $G$ be a group which acts on a topological space $M$ by homeomorphisms. **Then the natural projection $M \xrightarrow{\pi} M/G$ is an open map** (a map is **open** if it takes open sets to open sets).

**Example: the topological space of a graph**

**DEFINITION:** Let $\Gamma$ be a graph, and $S$ the set of its edges. Consider $S$ as the space with discrete topology, and let $X := S \times [0, 1]$ be a disconnected union of $S$ copies of an interval. For each $s \in S$, the points $s \times \{1\}$ and $s \times \{0\}$ corresponds to the ends of the intervals, each of them identified with the corresponding edge of the graph. If the edges $s_1$ and $s_2$ have a common vertex we write $x_1 \sim x_2$, where $x_i = s_i \times \{1\}$ or $x_i = s_i \times \{1\}$ are the corresponding points of $X$. **The topological space of a graph** is the quotient $X/\sim$.

**EXERCISE: The topological space of a graph is always Hausdorff (prove it).**

## Pseudo-metric spaces

**DEFINITION:** Let $M$ be a set. **A metric** on $M$ is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$, satisfying the following conditions.
* **[Symmetry:]** $d(x,y) = d(y,x)$,
* **[Triangle inequality:]** $d(x,y) \leqslant d(x,z) + d(z,y)$.
for any $x, y, z \in M$.
$d(x,x) = 0$ for all $x \in M$.
**REMARK:** Metrics are defined by three axioms, the definition of pseudo-metrics omit the first one:
* **[Non-degeneracy:]** $d(x,y) = 0 \Leftrightarrow x = y$.

**REMARK:** The condition $d(x,y) = 0$ defines an equivalence relation on $M$ **(prove it).**

**CLAIM:** Let $x \sim y$ be points in a pseudo-metric space $(M,d)$. **Then** $d(x,z) = d(y,z)$ for any $z \in M$.

**Proof:** Triangle inequality gives $d(x,z) \leqslant d(x,y) + d(y,z) = d(y,z)$, and similarly $d(y,z) \leqslant d(y,x) + d(x,z) = d(x,z)$. ■

**DEFINITION:** Let $x \in M$ be a point in a pseudo-metric space, and $\varepsilon \in \mathbb{R}^{\geqslant 0}$. The set $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ is called **an open ball** of radius $\varepsilon$ with center in $x$, or **an $\varepsilon$-ball**.

## Metric spaces as quotients of pseudo-metric spaces

**DEFINITION: An open set** in a pseudo-metric space $M$ is a union of open balls.

**REMARK: This topology is non-Hausdorff if $d$ is not a metric.** Indeed, for each points $x \sim y$, every open ball which contains $x$ also contains $y$, because $d(z,x) = d(z,y)$, hence $x$ **and** $y$ **cannot be separated**.

Let $(M,d)$ be a pseudo-metric space. Since $d(x,z) = d(y,z)$ for all $x \sim y$, the function $d$ is well defined on the space $\underline{M} := M/\!\sim$ of equivalence classes. Clearly, $d$ **defines a metric on the set** $\underline{M}$.

**CLAIM:** Every pseudo-metric space $(M,d)$ **is equipped with a surjective map** $\pi : M \longrightarrow \underline{M}$, where $(\underline{M}, \underline{d})$ is a metric space which satisfies

$$d(x,y) = \underline{d}(\pi(x), \pi(y)) \quad (*)$$

■

**REMARK:** Conversely, if $\pi : M \longrightarrow \underline{M}$ is a map from $M$ to a metric space, the formula (*) defines a pseudo-metric on $M$.

## Pseudo-metric on the quotient space

**DEFINITION:** Let $\sim$ be an equivalence relation on a metric space $(X, d)$. Define a function $d_\sim : X/\!\sim \times X/\!\sim \longrightarrow \mathbb{R}^{\geqslant 0}$ on the quotient $X/\!\sim$ using $d_\sim(x, y) = \inf \sum d(p_i, p_{i+1}) + d(q_{i+1}, q_{i+2})$, where the infimum is taken over all collections of points $p_i, q_i \in X$ such that $p_0 \sim x, q_n \sim y$, and $p_i \sim q_i$

**CLAIM: $d_\sim$ is a pseudo-metric on the quotient space $X/\!\sim$.**

**Proof:** We need only to prove the triangle inequality. However, $d_\sim$ is infimym of the lengths of the chains $p_0, p_1, q_1, q_2, p_2, p_3, q_3, q_4, ...$ connecting $x$ to $y$, where the distance between $p_i \sim q_i$ is set to 0. **If $x$ is connected to $y$, and $y$ to $z$ by such a chain, then $x$ is connected to $y$ by a concatenation of these two chains**, giving $d_\sim(x, z) \leqslant d_\sim(x, y) + d_\sim(y, z)$. ■

**DEFINITION:** Let $\sim$ be an equivalence relation on a metric space $(X, d)$. Then the pseudometric $d_\sim$ on $X/\!\sim$ is called **the quotient space metric**. **The metric quotient space** is obtained from $X/\!\sim$ by identifying all points $x, y$ which satisfy $d_\sim(x, y) = 0$.

**EXAMPLE:** Let $M$ be a group acting on a metric space $(X, d)$ by isometries, and $x \sim y$ if $x, y$ belong to the same $G$-orbit. **Then for any $a, b \in M/G$, the distance $d_\sim(a, b)$ is the infimum of the distance between the representatives of $a, b$ in $X$.**

## Metric graphs

**DEFINITION: Disconnected union** of metric spaces $(X_\alpha, d_\alpha)$ indexed by the index $\alpha$ is the union $\coprod X_\alpha$ with the metric $d(x,y)$ which is equal to $d_\alpha(x,y)$ when $x,y \in X_\alpha$, and to $\infty$ when $x \in X_\alpha, y \in X_\beta$ and $\alpha \neq \beta$.

**DEFINITION:** Let $I_\alpha$ be a collection of intervals, isometric to $[0, x_\alpha]$, and $\sim$ the equivalence relation, obtained by gluing of some vertices. The metric factor $\frac{\coprod_\alpha I_\alpha}{\sim}$ is called **the metric graph.** it is called **locally finite** if every point is identified with a finite number of points.

**REMARK:** When $\frac{\coprod_\alpha I_\alpha}{\sim}$ is locally finite, this space is homeomorphic to a topological space of a graph.

**CLAIM: The metric on a metric graph is always intrinsic.**

**Proof:** Every chain $p_0, p_1, q_1, q_2, p_2, p_3 ...$ connecting $x$ to $y$ can be realized by a connected union of intervals of the same length inside the graph. ■

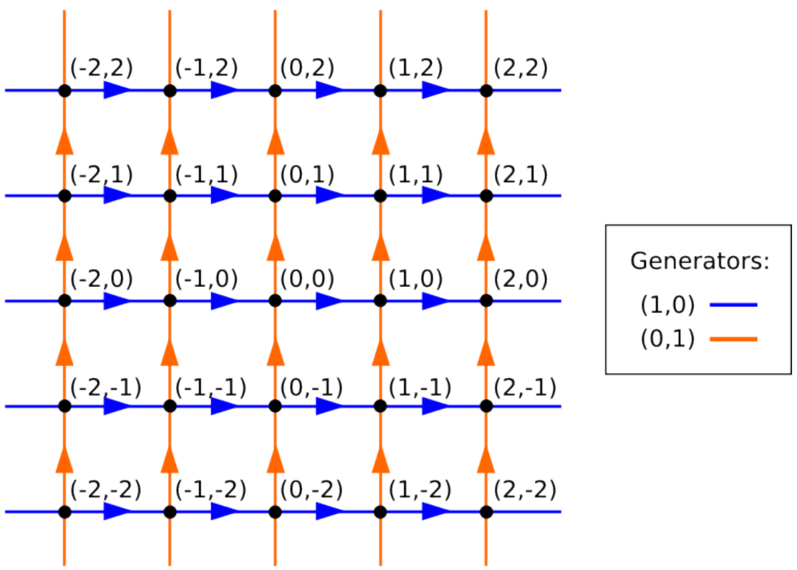
**A caution:** The natural map from the topological space of a graph $\Gamma$ to the metric graph **is a homeomorphism** for a locally finite graph. **It may fail to be bijective for a graph which is not locally finite.** Also, **it is never a homeomorphism, unless $\Gamma$ is locally finite.**

## Cayley graph

**DEFINITION: A set of generators** of a group $G$ is a set $S \subset G$ generating $G$ multiplicatively. **We would always assume that $s \in S \Leftrightarrow s^{-1} \in S$.**

**DEFINITION:** Let $G$ be a group, and $\{s_i\}$ a collection of generators. **The Cayley graph** of the pair $(G, \{s_i\})$ is the metric graph, with the set of vertices identified with $G$, and edges connecting $g$ and $gs_i$. The length of all edges the Cayley graph is set to the same number $t$, usually $t = 1$.

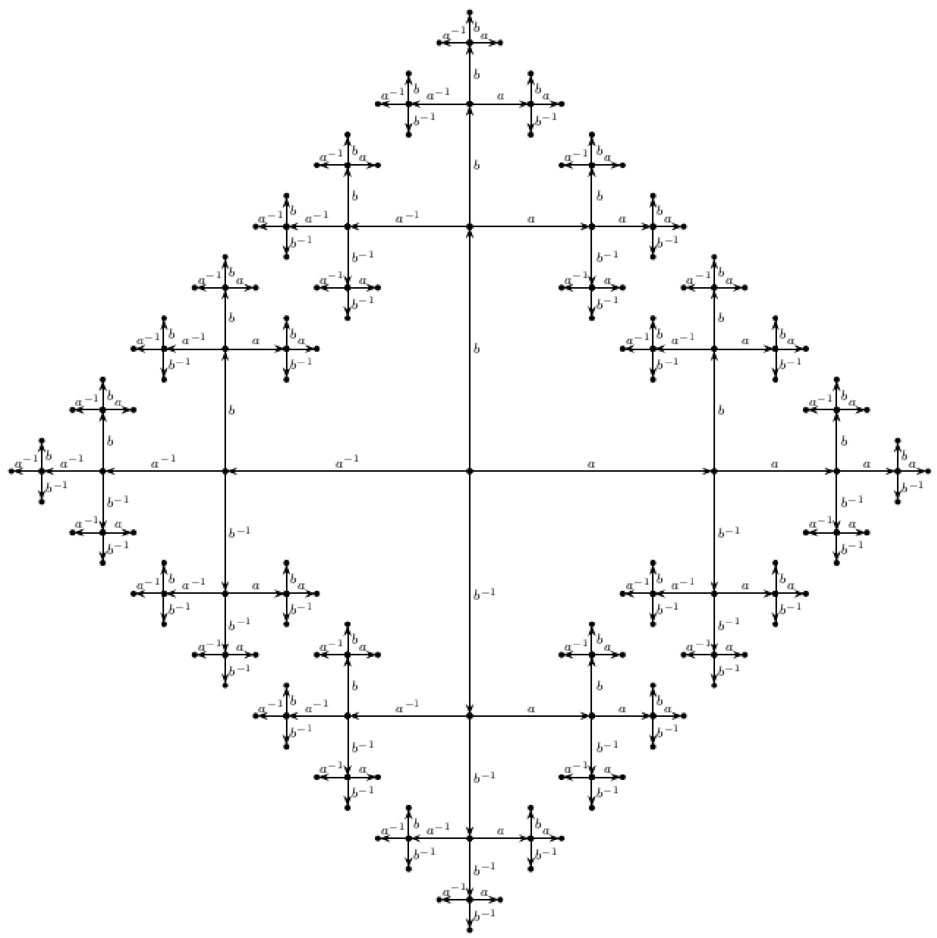
**EXAMPLE:** The Cayley graph for $\mathbb{Z}^n$ with the standard set of generators is a cubic lattice.



## Cayley graph for a free group

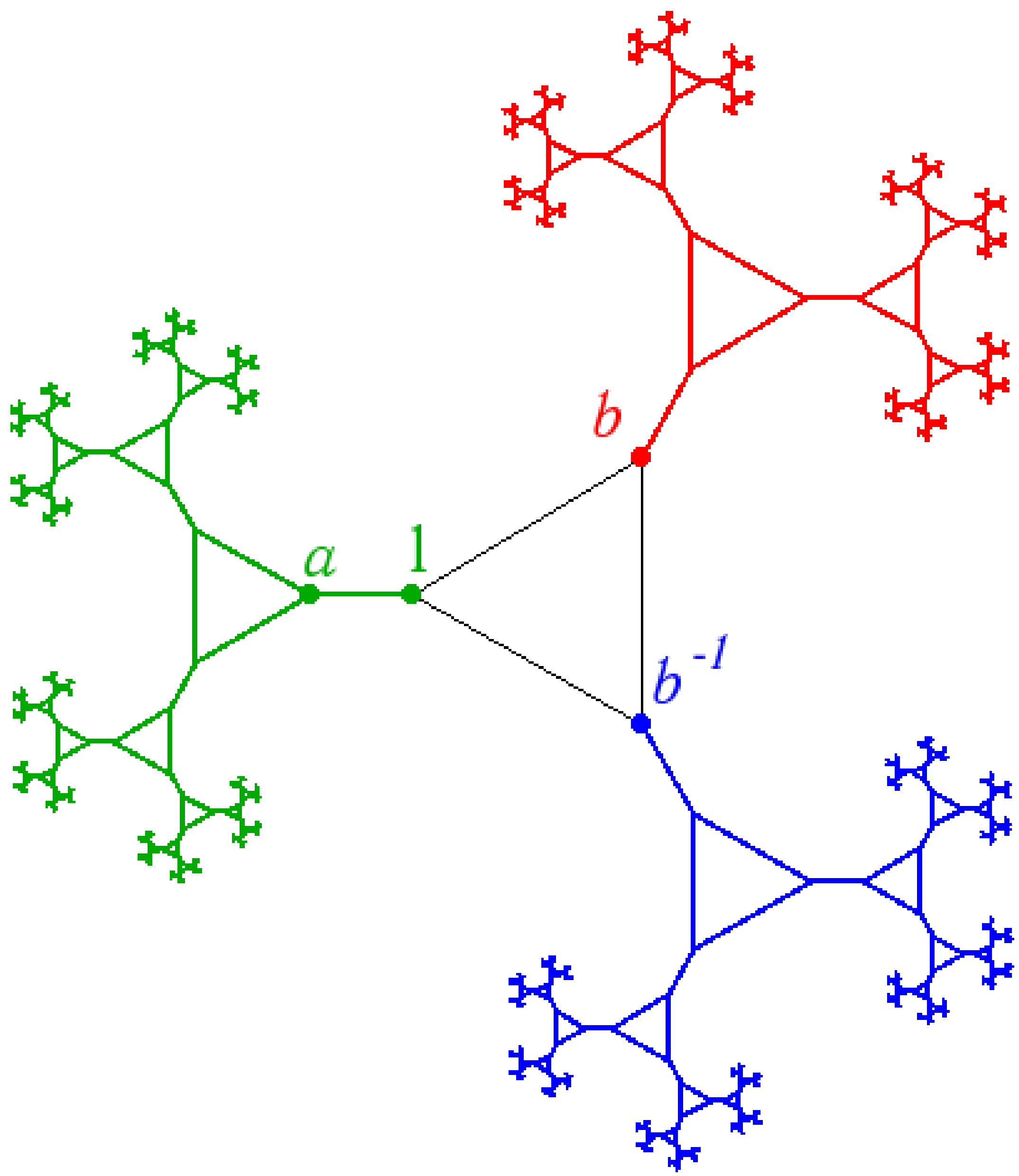
**REMARK:** Metric geometry of the Cayley graph **is the main subject of the geometric group theory.**

**EXAMPLE:** The Cayley graph for a free group is a regular tree,



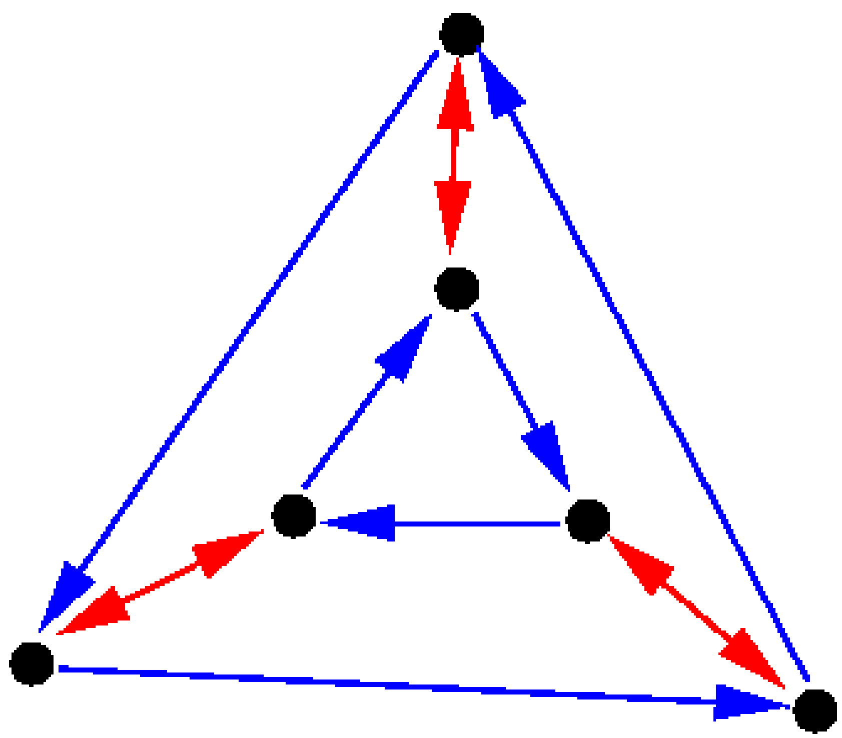
**Cayley graph for a free group $\mathbb{F}_2$ with generators $a$, $b$, $a^{-1}$, $b^{-1}$.**

## Cayley graph for $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$




**Cayley graph for the amalgamated product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.**

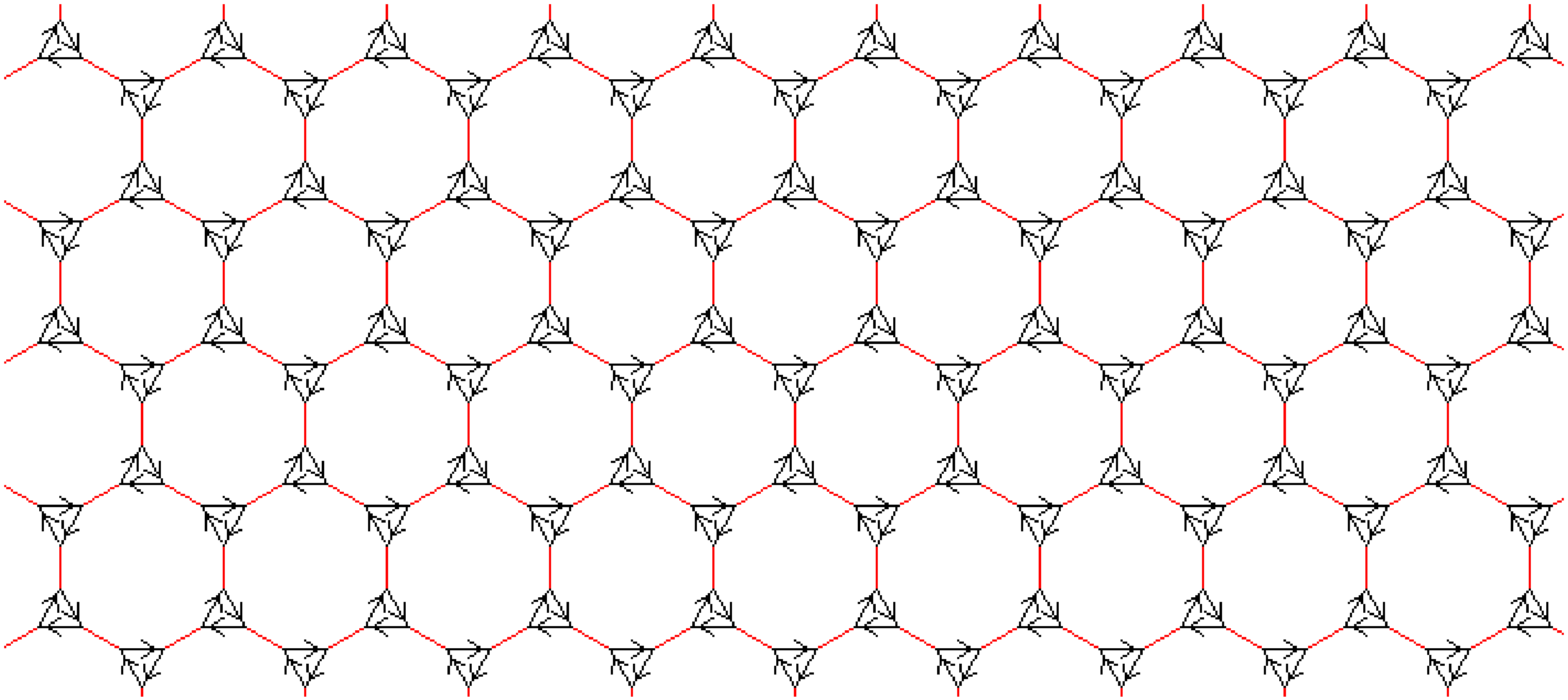
## Cayley graph for the symmetric group $\Sigma_3$



**Cayley graph for $\Sigma$.**

The symmetric group $\Sigma_3 = \langle k, r \mid k^2 = r^3 = (krk)^3 = 1\rangle$ is defined by a generator $k$ (red), $r$ (black), and relation $k^2 = r^3 = (krk)^3 = 1$.

## Cayley graph for the group $\langle k, r \mid k^2 = r^3 = (kr)^6 = 1 \rangle$



**Cayley graph for the group generated by $k$ (red), $r$ (black), and relations $k^2 = r^3 = (kr)^6 = 1$.**