Metric spaces

lecture 6: Polyhedral spaces

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Local metrics and weakly intrinsic metrics (reminder)

CLAIM: Let d_i be a family of metrics (possibly infinite), and $d(x,y) := \sup_i d_i(x,y)$. **Then d is also a metric.**

Proof: We need to check only that $d(x,y) \leqslant d(x,z) + d(z,y)$. This is clear, because

$$d(x,y) = \sup_i d_i(x,y) \leqslant \sup_i (d_i(x,z) + d_i(z,y)) \leqslant$$
$$\leqslant \sup_i d_i(x,z) + \sup_i d_i(z,y) = d(x,z) + d(z,y).$$

■

DEFINITION: Let $\{U_i\}$ be an open covering of s metric space $\{M,d\}$. Denote by $d_{\{U_i\}}$ the metric $\sup_\alpha d_\alpha$, where the supremum is taken over all metrics d_α which satisfy $d_\alpha\Big|_{U_i} = d$ for all open sets U_i in the cover. A metric d is called **$\{U_i\}$-local** if $d_{\{U_i\}} = d$. It is called **ε-local**, if it is $\{U_i\}$-local with respect to the covering $\{U_i\}$ consisting of all ε-balls, and **local** if it is ε-local for all $\varepsilon > 0$.

DEFINITION: For any two points x,y in a metric space (M,d), **an ε-chain**, connecting x to y is a collection of points $x = z_0, z_1, ..., z_{n-1}, z_n = y$ such that $d(z_i, z_{i+1}) \leqslant \varepsilon$. Its **defect** is the number $\sum_{i=0}^{n-1} d(z_i, z_{i+1}) - d(x,y)$. The space (M,d) is **weakly intrinsic** if for any two points $x,y \in M$ such that $d(x,y) < \infty$ and any $\varepsilon > 0$, $\delta > 0$, there exists an ε-chain connecting x to y with defect $\leqslant \delta$.

Hopf-Rinow theorem (reminder)

DEFINITION: For any two subsets $A, B \subset M$, we denote by $d(A, B)$ the number $\inf_{a\in A, b\in B} d(a, b)$.

DEFINITION: We say that a metric space (M, d) **admits ε-midpoints** if any $x, y \in M$ we have $d(B_x(r/2), B_y(r/2)) = 0$.

THEOREM: Let (M, d) be a metric space. **Then the following conditions are equivalent.**
(1). (M, d) is weakly intrinsic.
(2). (M, d) is local.
(3). For any $x, y \in M$, and any $r_1, r_2 > 0$ such that $d(x, y) = r_1 + r_2$, we have $d(B_x(r_1), B_y(r_2)) = 0$.
(4). (M, d) admits ε-midpoints.

Proof: Lecture 3. ■

THEOREM: **(Hopf-Rinow)**
Let M be a complete, locally compact space with a weakly intrinsic metric. **Then every closed metric ball $B_x^{cl}(r)$ in M is compact.**

Proof: Lecture 4. ■

Existence of geodesics (reminder)

DEFINITION: A continuous path $\gamma : [a,b] \longrightarrow M$ is called **minimizing**, if its arc-length is equal to $d(\gamma(a), \gamma(b))$.

DEFINITION: Let $\gamma : [a,b] \longrightarrow M$ be a path, and γ_1 takes $t \in \mathbb{R}^{\geqslant 0}$ to $\gamma(t)$, where t is the minimal of all $x \in [a,b]$ such that $L_d(\gamma\big|_{[a,x]}) = t$ (here, as everywhere, L_d denotes the arc length). When $t_i \in \mathbb{R}^{\geqslant 0}$ is the sequence converging to t, the sequence $\gamma(t_i)$ converges to t, because $L_d(\gamma_1\big|_{[t_i,t]}) \geqslant d(\gamma_1(t), \gamma_1(t_i))$. Therefore, γ_1 is continuous. Such a path γ_1 is called **the normal parametrization** of γ.

CLAIM: Let $\gamma : [a,b] \longrightarrow M$ be minimizing, and $\gamma_1 : [0, \alpha] \longrightarrow M$ its normal parametrization. **Then γ_1 is an isometry.**

DEFINITION: A **minimizing geodesic** is an isometry $\gamma : [a,b] \longrightarrow M$.

THEOREM: (Cohn-Vossen)
Let M be a locally compact, complete space with an almost intrinsic metric, and $x_0, x_1 \in M$. **Then there exists a minimizing geodesic, connecting x_0 to x_1.**

Metric quotient (reminder)

DEFINITION: Let $\sim$ be an equivalence relation on a metric space (X,d). Define a function $d_\sim : X/\!\sim \times X/\!\sim \longrightarrow \mathbb{R}^{\geqslant 0}$ on the quotient $X/\!\sim$ using $d_\sim(x,y) = \inf \sum d(p_i,p_{i+1}) + d(q_{i+1},q_{i+2})$, where the infimum is taken over all collections of points $p_i, q_i \in X$ such that $p_0 \sim x, q_n \sim y$, and $p_i \sim q_i$

CLAIM: $d_\sim$ is a pseudo-metric on the quotient space $X/\!\sim$.

Proof: We need only to prove the triangle inequality. However, $d_\sim$ is infimym of the lengths of the chains $p_0, p_1, q_1, q_2, p_2, p_3, q_3, q_4, ...$ connecting x to y, where the distance between $p_i \sim q_i$ is set to 0. **If x is connected to y, and y to z by such a chain, then x is connected to y by a concatenation of these two chains**, giving $d_\sim(x,z) \leqslant d_\sim(x,y) + d_\sim(y,z)$. ■

DEFINITION: Let $\sim$ be an equivalence relation on a metric space (X,d). Then the pseudometric $d_\sim$ on $X/\sim$ is called **the quotient space metric**. **The metric quotient space** is obtained from $X/\sim$ by identifying all points x, y which satisfy $d_\sim(x,y) = 0$.

EXAMPLE: Let M be a group acting on a metric space (X,d) by isometries, and $x \sim y$ if x, y belong to the same G-orbit. **Then for any $a, b \in M/G$, the distance $d_\sim(a,b)$ is the infimum of the distance between the representatives of a, b in X.**

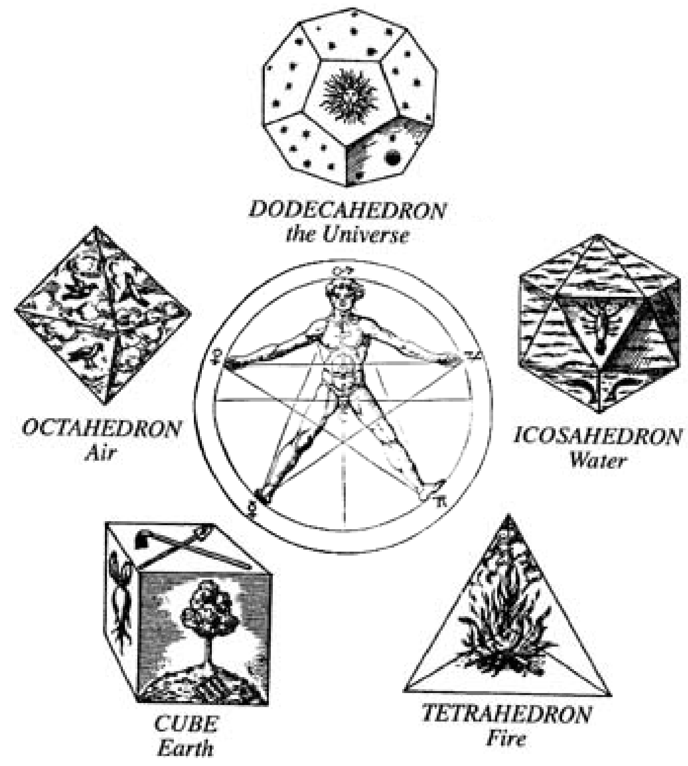
Intrinsic metric and metric gluing

DEFINITION: Let $M_1, ..., M_n$ be a collection of metric spaces with weakly intrinsic metric, and $Z_{ij} \subset M_i$ is a collection of metric subsets. Assume that the restriction of the metric from M_i to Z_{ij} is also weakly intrinsic. Fix a collection of isometries $\Psi^{ij}_{kl} : Z_{ij} \longrightarrow Z_{kl}$. Consider a quotient of $M := \frac{\coprod M_i}{\sim}$ by the equivalence relation generated by $x \sim \Psi^{ij}_{kl}(x)$. We say that M **is a metric space obtained from the union of M_i by gluing $Z_{ij} \subset M_I$ to $Z_{kl} \subset M_k$ along Ψ^{ij}_{kl}.**

THEOREM: The metric on M obtained by gluing is weakly intrinsic.

Proof: By definition, $d_M(x, y)$ is infimum of the length of the chains $p_0, p_1 \in M_{k_0}, q_1, q_2 \in M_{k_1}, p_2, p_3 \in M_{k_1}, ...$ where p_i is glued to q_i. The defect of this chain is equal to $\delta := -d_M(x, y) + \sum d_{M_{k_i}}(p_i, p_{i+1}) + d_{M_{k_{1+1}}}(q_{i+1}, q_{i+2})$, which can be chosen smaller than any given number $\delta' > 0$. If we choose ε-chains in M_i connecting p_i to p_{i+1} and q_i to q_{i+1} with sufficiently small defect δ_i, this would give us an ε-chain connecting x to y with defect $\delta + \sum \delta_i$, which can be chosen arbitrarily small. This gives an ε-chain connecting x to y with arbitrarily small defect. ■

Platonic solids



Pythagorean Cosmic Morphology

Convex polyhedra

DEFINITION: A closed convex polyhedron in $\mathbb{R}^n$ is an intersection of finitely many closed half-spaces, that is, subsets of $\mathbb{R}^n$ isometric to $\mathbb{R}^{n-1}\times\mathbb{R}^{\geqslant 0}$. It is called **bounded** if it is compact. It is called **n-dimensional** if its interior is non-empty. We consider a convex polyhedron as a metric space, with the metric induced from $\mathbb{R}^n$. Clearly, **this metric is intrinsic.**

REMARK: A boundary ∂P of a polyhedron P is clearly a union of polyhedra of smalled dimension. However, the metric on P restricted on ∂P is not intrinsic, because the geodesics in P with the ends in ∂P don't generally belong to P. **The intrinsic metric in ∂P** is the metric where $d(x,y)$ is infimum of the arc-length of all paths in ∂P connecting x to y.

DEFINITION: Suppose that P is an n-dimensional polyhedron which belongs to a half-space H, and $\partial P\cap\partial H$ has dimension $n-1$. Then $\partial P\cap\partial H$ is called **a face** of P.

Intrinsic metric on a boundary of a polyhedron

EXERCISE: Prove that **every face of an n-polyhedron is an $n-1$-dimensional convex polyhedron,** and ∂P **is a union of all faces of P.**

CLAIM: The space ∂P with its intrinsic metric **is obtained by gluing all its faces over their pairwise intersections.**

Proof: A path γ in ∂P is a union of paths which belong to its faces. Since each face is convex, we can replace each of these paths by a straight segment I_i within each face. Then $L_d(\gamma)$ is bounded by $\sum_i |I_i|$ which is equal to a length of the chain $p_0, p_1, q_1, q_2, ...$ where each I_i is an interval with ends in p_i, p_{i+1} or q_i, q_{i+1}. Conversely, any such chain corresponds to a polygonal chain of the same length, hence the metric in ∂P obtained by gluing of faces coincides with the intrinsic metric. ■

REMARK: ∂P **is an example of a polyhedral metric space** which I am going to define in the next slide.

Polyhedral metric spaces

DEFINITION: A polyhedral metric space of dimension 1 is a metric graph.

DEFINITION: A polyhedral metric space of dimension k is defined inductively as follows. Every k-dimensional metric space K is obtained by gluing its **l-skeletons** K_l, $l = 1, 2, 3, ..., k$, which are polyhedral metric spaces of dimension l. The space K_k is obtained from K_{k-1} by gluing K_{k-1} to a collection of convex polyhedra in $\mathbb{R}^k$, as follows.

Let K be a polyhedral metric space of dimension $k-1$ and $V_1, ..., V_n$ be a collection of convex, bounded, closed k-dimensional polyhedra. For every V_i we fix a closed embedding $\tau_i : \partial V_i \longrightarrow K_{k-1}$ from its boundary to K_{k-1}. Assume that τ_i is an isometry on every face of ∂V_i.

A polyhedral metric space of dimension k is a space obtained by gluing the k-dimensional polyhedra V_i to a polyhedral metric space of dimension $k-1$, denoted K_{k-1}, using a a map $\tau_i : \partial V_i \longrightarrow K_{k-1}$ which is isometric on each face of V_i. We assume that K **is locally finite,** that is, every point of the skeleta K_l belongs to only finitely many polyhedra used in this construction.

REMARK: This is the model example of an intrinsic metric space used in metric geometry.