Metric spaces

lecture 7: Angles and cones

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Angles

REMARK: In the sequel, **every time I mention $\mathbb{R}^n$, I consider it as a metric space with the Euclidean distance metric.**

DEFINITION: Let a, b, c be points in a metric space (M, d). **A comparizon triangle** $\triangle(\overline{a}, \overline{b}, \overline{c})$ is a triangle in $\mathbb{R}^2$, with vertices $\overline{a}, \overline{b}, \overline{c}$, and side lengths $|\overline{a}, \overline{b}| = d(a, b)$, $|\overline{a}, \overline{c}| = d(a, c)$, $|\overline{b}, \overline{c}| = d(b, c)$. **This triangle exists, and is uniquely determined, up to an isometry,** (the existence follows from the triangle inequality). The angle $\measuredangle(\overline{a}, \overline{b}, \overline{c}) \in [0, \pi]$ in the triangle $\overline{a}, \overline{b}, \overline{c}$ is denoted $\theta(a, b, c)$; it is called **the comparizon angle**.

DEFINITION: Let $\gamma_1 : [0, a] \longrightarrow M$, $\gamma_2 : [0, b] \longrightarrow M$ two paths in a metric space M, with $\gamma_1(0) = \gamma_2(0) = p$. **The angle** between the paths γ_1, γ_2 is the number

$$\measuredangle(\gamma_1, p, \gamma_2) := \lim_{t,s \to 0} \theta(\gamma_1(t), p, \gamma_2(s)),$$

if the limit is defined (otherwise, we say that the angle does not exist). The **upper angle** is

$$\measuredangle_{\sup}(\gamma_1, p, \gamma_2) := \limsup_{t,s \to 0} \theta(\gamma_1(t), p, \gamma_2(s)).$$

EXERCISE: Prove that **the angle between smooth paths in $\mathbb{R}^n$ exists and is equal to the angle between their tangents**.

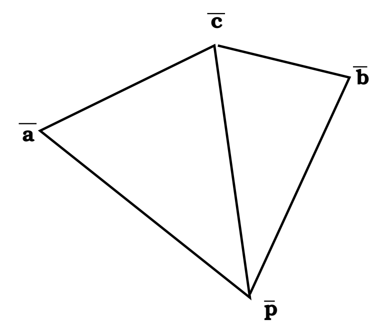
Triangle inequality for upper angles

EXERCISE: Let $\gamma : : [0,a] \longrightarrow M$ be a minimizing geodesic, and $\gamma(0) = p$. Prove that **the angle $\measuredangle(\gamma, p, \gamma)$ exists and is equal to zero.**

THEOREM: Let $\gamma_i : [0,a] \longrightarrow M$, $i = 1,2,3$ be paths in M, $\gamma_i(0) = p$. **Then the following version of the triangle inequality holds:**

$$\measuredangle_{\sup}(\gamma_1, p, \gamma_2) + \measuredangle_{\sup}(\gamma_2, p, \gamma_3) \geqslant \measuredangle_{\sup}(\gamma_1, p, \gamma_3).$$

Proof. Step 1: To simplify the notation, we parametrize γ_i in such a way that $d(p, \gamma_i(x)) = x$. Let $a = \gamma_1(s), b = \gamma_3(t)$, $c = \gamma_2(u)$. Consider the comparizon triangles $\triangle(\overline{p}, \overline{a}, \overline{c})$ and $\triangle(\overline{p}, \overline{c}, \overline{b})$, and draw them on the plane, on different sides of the line $(\overline{p}, \overline{c})$.



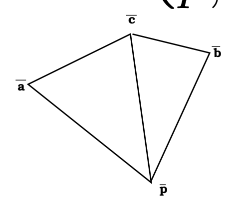
Since a point $\overline{x}$ in the triangle $\triangle(\overline{p}, \overline{a}, \overline{b})$ with sides s and t satisfies $d(\overline{p}, \overline{x}) < s + t$, for any $u > s + t$, the point $\overline{c}$ lies outside the triangle $\triangle(\overline{p}, \overline{a}, \overline{b})$. Either the quadrilateral $(\overline{p}, \overline{a}, \overline{c}, \overline{b})$ remains convex for all u, or it stops being convex for u sufficiently small. **In this case, by continuity, for some $u > 0$, $\overline{c}$ lies on the interval $[\overline{a}, \overline{b}]$.**

Triangle inequality for upper angles (2)

THEOREM: Let $\gamma_i : [0,a] \longrightarrow M$, $i=1,2,3$ be paths in M, $\gamma_i(0)=p$. **Then the following version of the triangle inequality holds:**

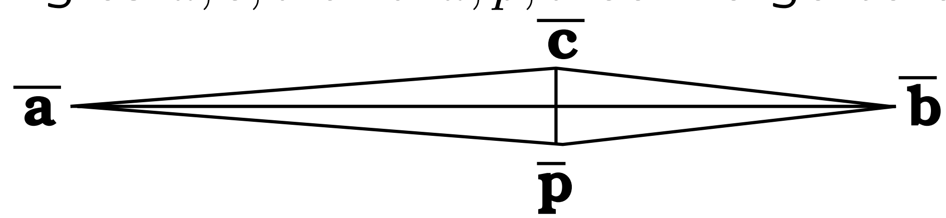
$$\measuredangle_{\sup}(\gamma_1,p,\gamma_2)+\measuredangle_{\sup}(\gamma_2,p,\gamma_3)\geqslant\measuredangle_{\sup}(\gamma_1,p,\gamma_3).$$

Proof. Step 1: (abbreviated)
Consider the comparizon triangles $\triangle(\overline{p},\overline{a},\overline{c})$ and $\triangle(\overline{p},\overline{c},\overline{b})$, and draw them on the plane, on different sides of the line $(\overline{p},\overline{c})$.



Then quadrilateral $(\overline{p},\overline{a},\overline{c},\overline{b})$ remains convex for all u, or it stops being convex for u sufficiently small.

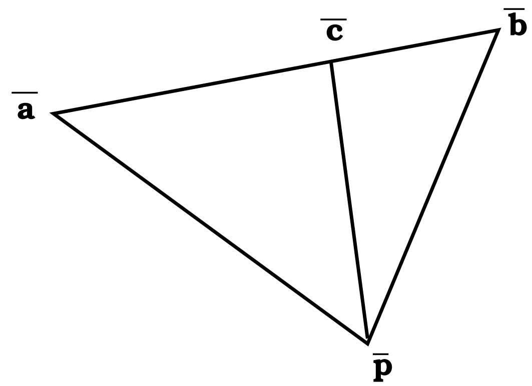
Step 2: We prove that in the first case, the triangle inequality is trivial. When $u\to 0$, the triangles $\overline{a},\overline{c},\overline{b}$ and $\overline{a},\overline{p},\overline{b}$ converge to a line:



This implies, in particular, the sum of angles in $\overline{p}$ converges to π. Then $\measuredangle_{\sup}(\gamma_1,p,\gamma_2)+\measuredangle_{\sup}(\gamma_2,p,\gamma_3)\geqslant\pi\geqslant\measuredangle_{\sup}(\gamma_1,p,\gamma_3)$, because **in our definition no angle can be bigger than π.**

Triangle inequality for upper angles (2)

Step 3: Consider the comparizon triangles for $\overline{c}$ on the line connecting $\overline{b}$ and $\overline{a}$.



Then

$$\theta(a,p,c)+\theta(c,p,b)=\measuredangle(\overline{a},\overline{p},\overline{b})=\operatorname{arccos}\left(\frac{s^2+t^2-|\overline{a},\overline{b}|^2}{2st}\right) \quad (*)$$

where $s=d(p,a)$ and $t=d(p,b)$.

Step 4: By definition, $|\overline{a},\overline{c}|=|\overline{a},\overline{b}|+|\overline{b},\overline{c}|=d(a,b)+d(b,c)\geqslant d(a,c)$. Monotonicity of arccosine, together with $|\overline{a},\overline{b}|=d(a,c)+d(c,b)\geqslant d(a,b)$ gives

$$\measuredangle(\overline{a},\overline{p},\overline{b})=\operatorname{arccos}\left(\frac{s^2+u^2-|\overline{a},\overline{b}|^2}{2su}\right)\geqslant \operatorname{arccos}\left(\frac{s^2+u^2-d(a,b)^2}{2su}\right)=\theta(a,p,b). \quad (**)$$

Step 5: Comparing (*) and (**) we obtain $\theta(a,p,c)+\theta(c,p,b)\geqslant\theta(a,p,b)$; the triangle inequality for $\measuredangle_{\text{sup}}$ follows. ■

The space of directions

DEFINITION: We say that a path $\gamma : [0, a] \longrightarrow M$ **has a direction** if the angle $\measuredangle(\gamma, \gamma(0), \gamma)$ exists. The paths $\alpha, \beta : [0, a] \longrightarrow M$, $\alpha(0) = \beta(0) = p$ **have the same direction** if $\measuredangle_{\sup}(\alpha, p, \beta) = 0$.

EXAMPLE: A minimizing geodesic $\gamma : [0, a] \longrightarrow M$ has a direction. Indeed, for any reals $t_2 > t_1 > 0$, we have $d(\gamma(0), \gamma(t_2)) = d(\gamma(0), \gamma(t_1)) + d(\gamma(t_1), \gamma(t_2))$, hence all sides of the corresponding comparizon triangle are collinear. **This gives $\measuredangle(\gamma(t_1), \gamma(0), \gamma(t_2)) = 0$.**

REMARK: Given two paths $\alpha, \beta : [0, a] \longrightarrow M$, $\alpha(0) = \beta(0) = p$ which have a direction, we write $\alpha \sim \beta$ if $\measuredangle_{\sup}(\alpha, p, \beta) = 0$. Triangle inequality for upper angles, implies that σ defines an equivalence relation.

DEFINITION: The space of directions in $b \in M$ is the set of $\sim$-equivalence classes of paths $\gamma : [0, a] \longrightarrow M$, $\gamma(0) = p$ which have directions.

CLAIM: The function $\alpha, \beta \mapsto \measuredangle_{\sup}(\alpha, p, \beta)$ **defines a metric on the space of directions.** ■

The cone

DEFINITION: The diameter $\operatorname{diam}(M)$ of a metric space M is the number $\sup_{x,y\in M} d(x,y)$.

DEFINITION: Let (X,d) be a metric space, $\operatorname{diam} X \leqslant \pi$. Consider the topological space $C(X)$ with the quotient topology, obtained from $X \times [0,\infty[$ by gluing $X \times \{0\}$ into a single point, called **the origin**, or **the apex**, and $C(X)$ **the cone over X**. Define the function $d_C: C(X) \times C(X) \longrightarrow \mathbb{R}^{>0}$ as

$$d(p,q) = \sqrt{t^2 + s^2 - 2ts\cos(d(x,y))},$$

where $p = (x,t), q = (y,s)$.

REMARK: Soon enouth we shall see that d_C is a metric. It is called **the cone metric**, and $C(X)$ **the metric cone.**

REMARK: This definition has the following geometric meaning. Let $\gamma : [0, d(x,y)] \longrightarrow X$ be a minimizing geodesic connecting x to $y \in X$. Draw a geodesic A of the same length on a unit circle in $\mathbb{R}^2$. Its cone $C(A)$ is a circular sector in $\mathbb{R}^2$, obtained as a cone over A. **We define the distance in $C(A)$ in such a way that the embedding to $\mathbb{R}^2$ is an isometry.**

A cone over a triangle is a polyhedral metric space

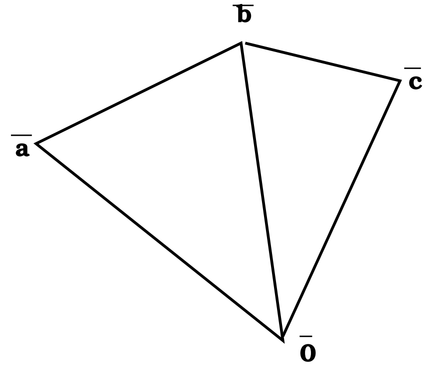
REMARK: The thriangle inequality **is very easy to see for the cones over geodesic spaces.** When any two points in X are connected by minimizing geodesics, any two points in $C(X)$ belong to a cone over an interval which is isometric to a polyhedral domain in $\mathbb{R}^2$ bounded by two rays. Any three points in $C(X)$ are situated on a polyhedral space, obtained by gluing three such polyhedral domains over their boundaries, and **the triangle inequality in $C(X)$ follows from the one in this polyhedral metric space.**

Triangle inequality for the cone metric

THEOREM: The function d_C satisfies the triangle inequality.

Proof. Step 1: Let $(\alpha,t),(\beta,s)$ be points in the cone $C(X)$, and $\triangle(\overline{0},\overline{a},\overline{b})$ the comparison triangle with sides t,s and the angle $\measuredangle(\overline{a},\overline{0},\overline{b})=d(\alpha,\beta)$. **Then $d_C(a,b)=|\overline{a},\overline{b}|$.**

Step 2: Let $a=(\alpha,r), b=(\beta,s), c=(\gamma,t)$ be points on $C(X)$, and $\triangle(\overline{0},\overline{a},\overline{b})$, $\triangle(\overline{0},\overline{b},\overline{c})$ the corresponding comparison triangles, with a common side $[\overline{0},\overline{b}]$, drawn on different sides of the interval $(\overline{0},\overline{b})$.



By definition, $d_c(a,c)=\sqrt{r^2+t^2-2rt\cos(d(\alpha,\beta))}$ and

$$|\overline{a},\overline{c}|=\sqrt{r^2+t^2-2rt\cos(\measuredangle(\overline{a},\overline{0},\overline{b})+\measuredangle(\overline{b},\overline{0},\overline{c}))}\geqslant\sqrt{r^2+t^2-2rt\cos(d(\alpha,\beta))}$$

because $\measuredangle(\overline{a},\overline{0},\overline{b})=d(\alpha,\beta)$ and $\measuredangle(\overline{b},\overline{0},\overline{c})=d(\beta,\gamma)$. Then $d_C(a,c)\leqslant|\overline{a},\overline{c}|\leqslant|\overline{a},\overline{b}|+|\overline{b},\overline{c}|=d_C(a,b)+d_C(b,c)$. ■

Properties of the cone

PROPERTIES OF THE METRIC CONE:
1. For any $x \in X$, **the path** $\gamma : [0, a] \longrightarrow C(X)$**, taking** a **to** (x, a) **is a minimizing geodesic.**

2. Let $x, y \in X$, and let $\gamma_1 := (x, [0, a])$, $\gamma_2 := (y, [0, b]) \subset C(X)$ be the corresponding geodesic in the cone. **Then** $\measuredangle(\gamma_1, 0, \gamma_2) = d(x, y)$**.**

3. **A cone over the interval of length** α **is isometric to a circular sector of angle** α**.**

CLAIM: Let X be a space with interior metric and minimizing geodesics. **Then the metric in** $C(X)$ **is also interior and has minimizing geodesics.**

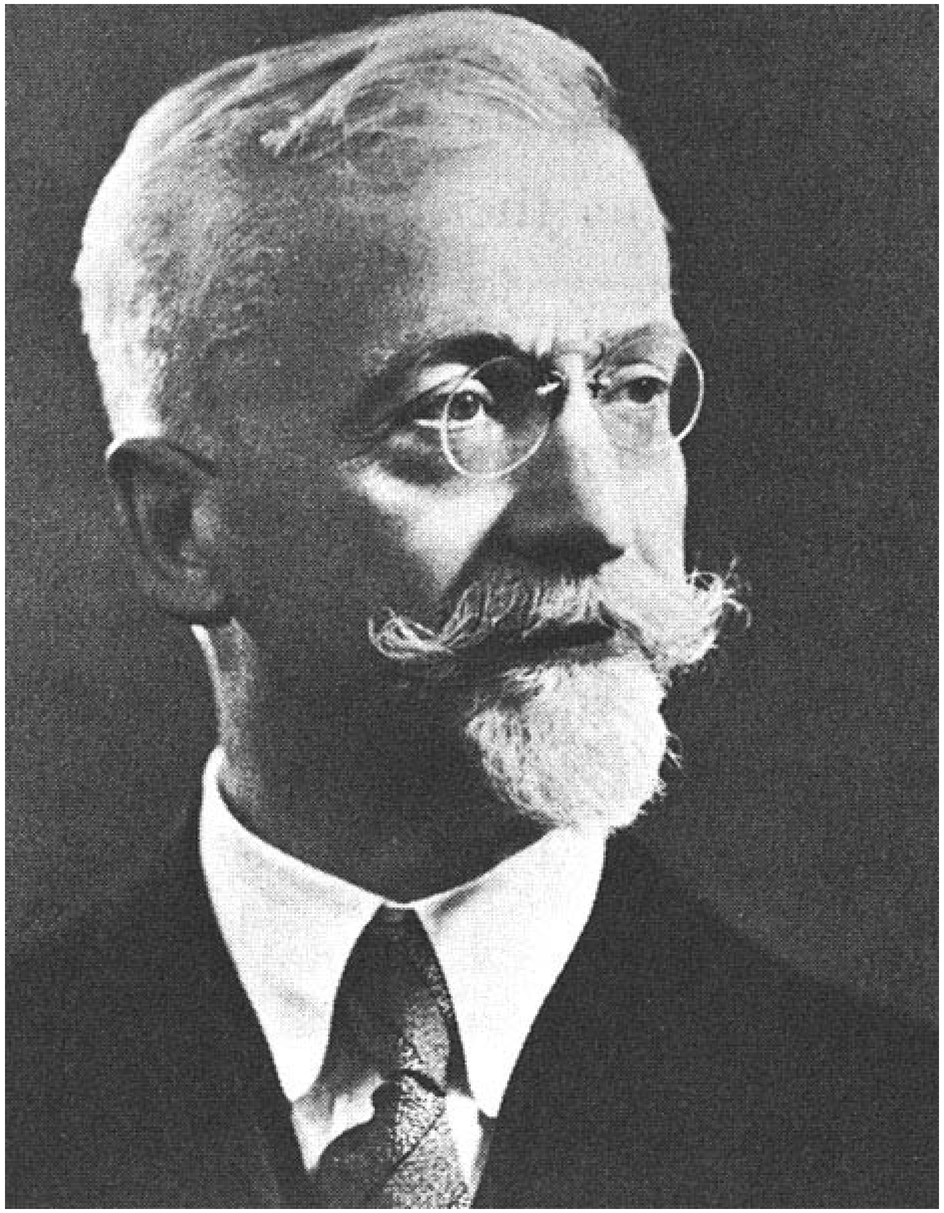
Proof: For any minimizing geodesic $\gamma \subset X$, the cone $C(\gamma)$ **is isometric to a circular sector of angle** α**.** Then any two points (a, s) and (b, t) can be realized as points on a circular sector $C(\gamma)$, isometrically embedded to $C(X)$, and they can be connected by a minimizing geodesic within $C(\gamma)$. ■

Alexandrov spaces

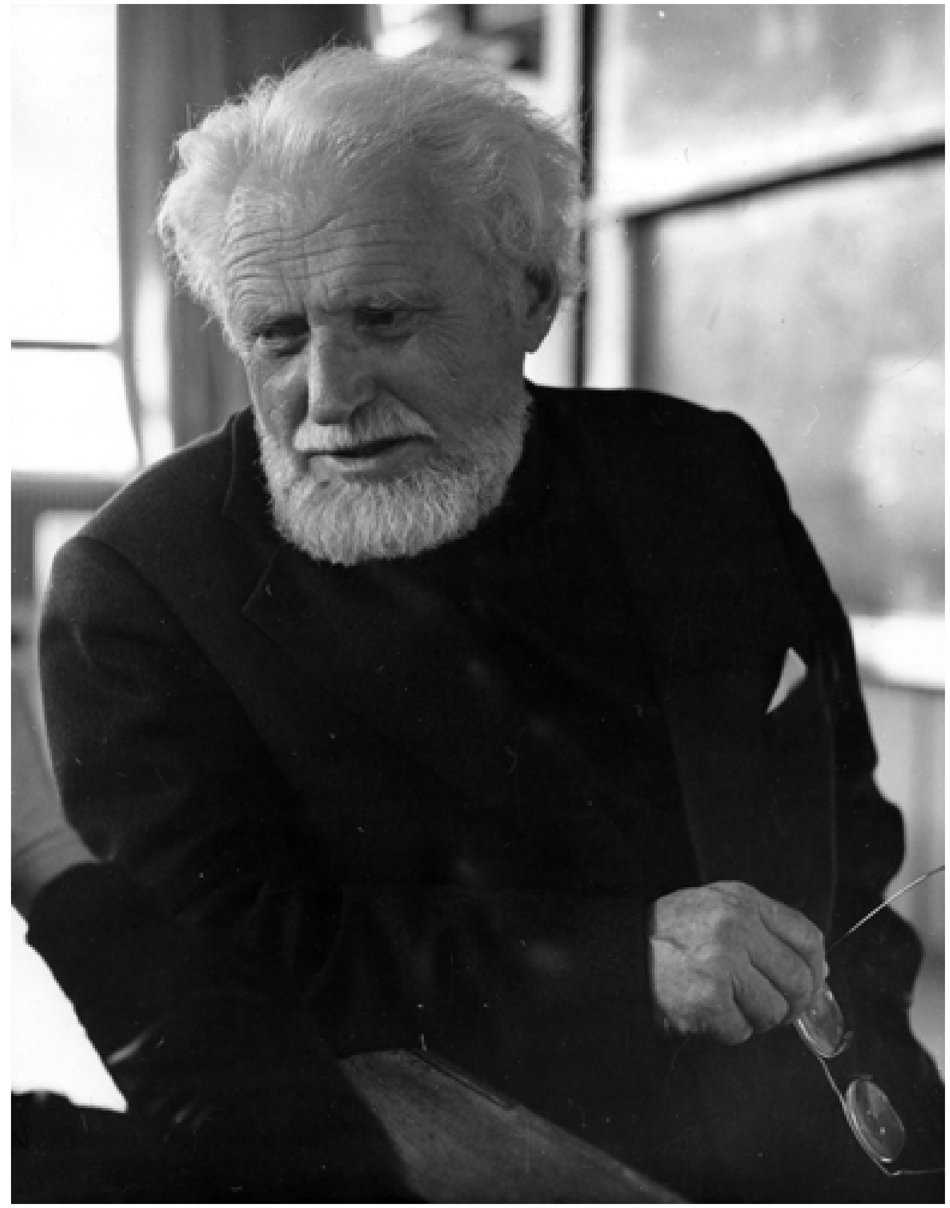
DEFINITION: Let a, b, c be points on a metric space (M, d) with geodesic metric, and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An intrinsic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for any geodesic triangle.

REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

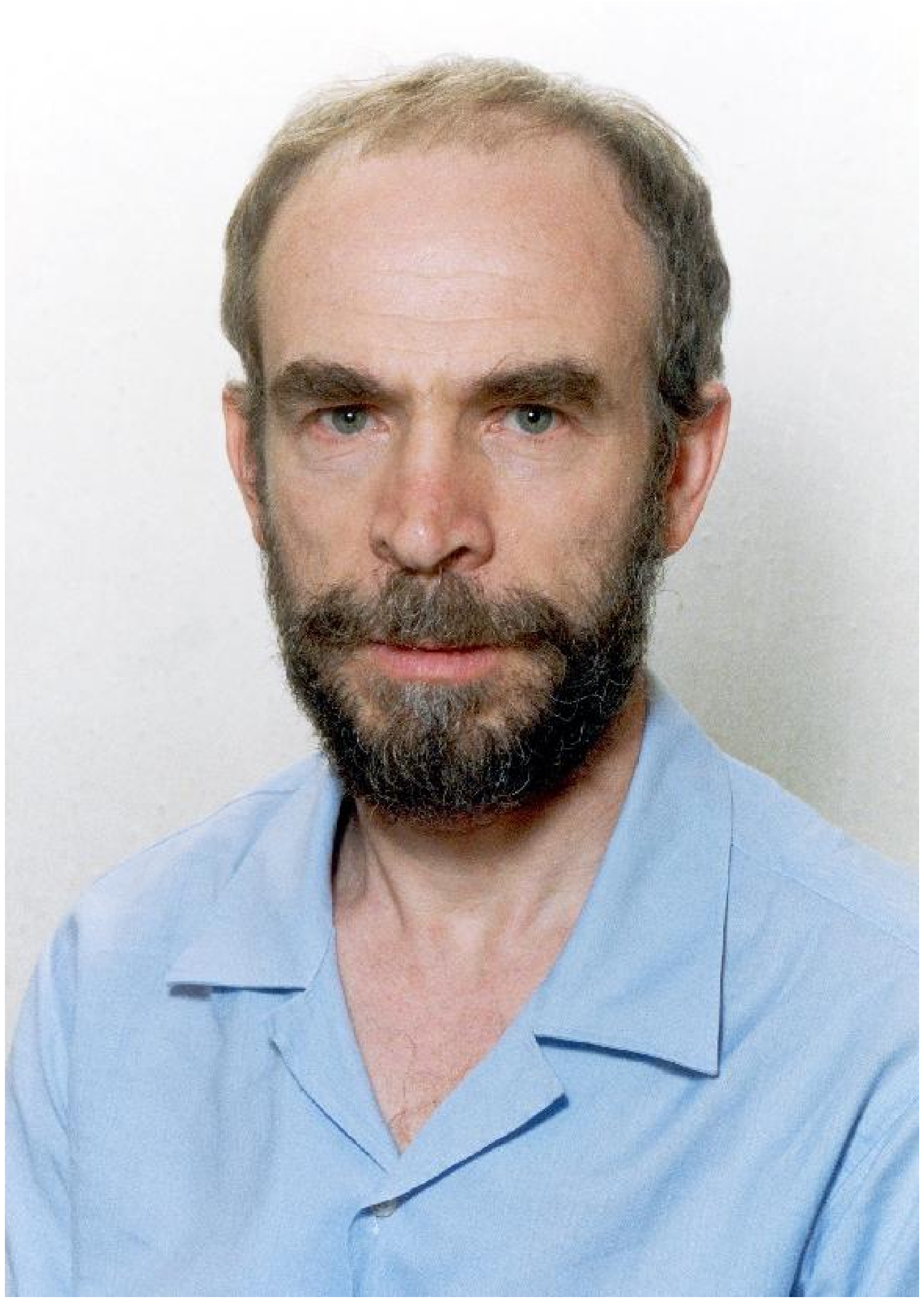


Élie Cartan,
1869-1951



Alexander Danilovich Alexandrov,
1912-1999

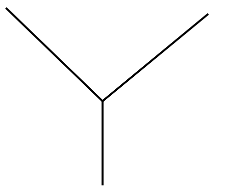
Viktor Andreevich Toponogov,
1930-2004



Misha Gromov
(b. 23 Dec. 1943)

Examples of Alexandrov spaces

EXAMPLE: Let Z be a metric graph.



Then Z is a space of non-positive curvature.

EXAMPLE: Let L be a circle of length $d \leqslant 2\pi$ with interior metric, and $C(L)$ is cone. **Then $C(L)$ is a space of non-negative curvature.**

EXAMPLE: **A notebook** is a polyhedral space of dimension 2, with the quotient metric, obtained by gluing several half-planes over the boundary line. **The notebook is a CAT(0)-space.**

EXAMPLE: **A metric bouquet** of spaces M_i with marked point x_i is obtained from these spaces by gluing the points x_i together (we put the quotient metric on it). **A metric bouquet of spaces of non-positive curvature has non-positive curvature.**

EXERCISE: **Prove all these assertions.**