Metric spaces

lecture 8: Alexandrov spaces

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Alexandrov spaces (reminder)

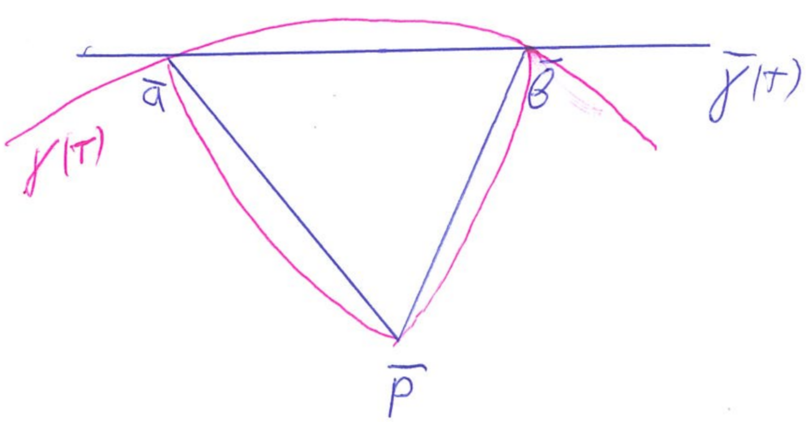
DEFINITION: Let a, b, c be points on a metric space (M, d) with geodesic metric, and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An intrinsic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for all geodesic triangles.

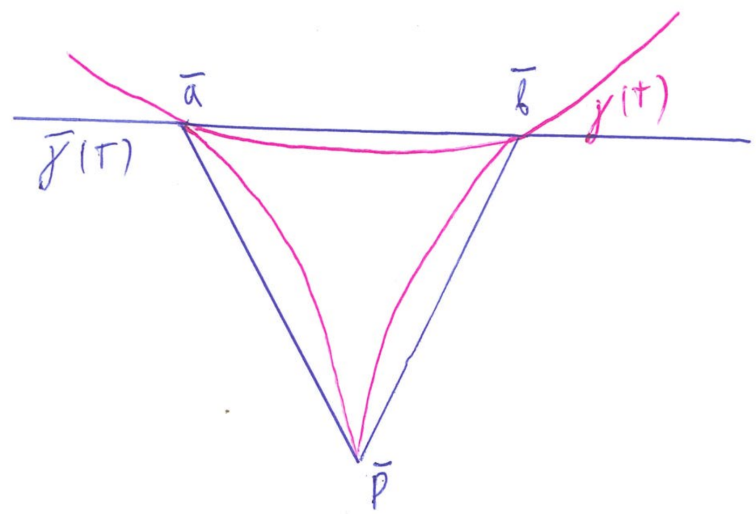
REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

Alexandrov conditions and functions on the whole geodesic

PROPOSITION: Let $\gamma : [x, y] \longrightarrow M$ be a minimizing geodesic. Choose distinct points $a = \gamma(u), b = \gamma(v)$ on $\operatorname{im}\gamma$, and let $p \neq \gamma$. Consider the comparison triangle $\triangle(\overline{a}\overline{b}\overline{p}) \subset \mathbb{R}^2$. Consider an interval of a line $(\overline{a}, \overline{b}) \subset \mathbb{R}^2$ normally parametrized by a map $\overline{\gamma} : [x, y] \longrightarrow \mathbb{R}^2$, in such a way that $\overline{a} = \overline{\gamma}(u), \overline{b} = \overline{\gamma}(v)$. Let $d_p : [x, y] \longrightarrow \mathbb{R}^{\geqslant 0}$ take $t \in [x, y]$ to $d(\gamma(t), p)$ and $d_{\overline{p}} : [x, y] \longrightarrow \mathbb{R}^{\geqslant 0}$ take $t \in [x, y]$ to $|\overline{\gamma}(t), \overline{p}|$. **Then $d_{\overline{p}} \geqslant d_p$ on $[x, y] \backslash [u, v]$ if the non-positive curvature Alexandrov condition $d_{\overline{c}} \geqslant d_c$ holds everywhere M, and $d_{\overline{p}} \leqslant d_p$ on $[x, y] \backslash [u, v]$ if the non-negative curvature Alexandrov condition $d_{\overline{c}} \leqslant d_c$ holds on the whole of M** (for all geodesic triangles)

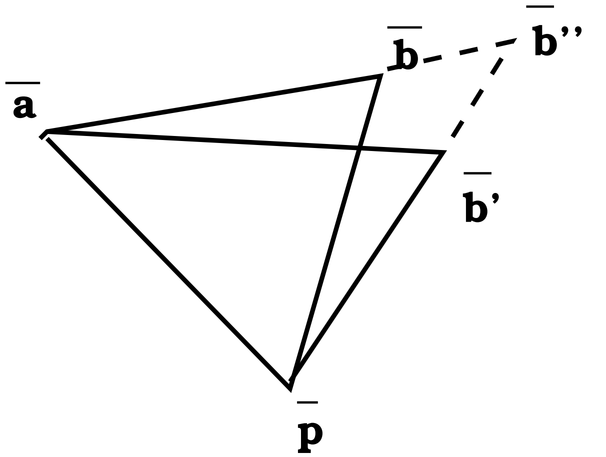


non-negative curvature



non-positive curvature

Proof: Assume that the non-positive curvature Alexandrov condition holds everywhere on M, but for some $t \in [x, y]\backslash[u, v]$, one has $d_{\overline{p}}(t) < d_p(t)$. Assume that $t > v$; the case $t < u$ is treated the same way. Let $b' := \gamma(t)$, and let $\triangle(\overline{a}, \overline{p}, \overline{b}')$ be the comparizon triangle. Denote by $d'_{\overline{p}}$ the distance to $\overline{p}$ in $\mathbb{R}^2$ considered as a function on the interval $[u, t]$ identified with $|\overline{a}, \overline{b}'|$. The inequality $d(\overline{p}, \overline{b}') < d(\overline{p}, \overline{b})$ implies $d'_{\overline{p}} < d_{\overline{p}}$ on $]u, t]$, as seen from the picture below. Then, on $]u, v] \subset]u, t]$, we have $d_{\overline{p}} > d'_{\overline{p}} \geqslant d_p$, implying a contradiction $d(p, b) = d_{\overline{p}}(v) > d'_{\overline{p}}(v) \geqslant d_p(v)$ (the last inequality comes from the non-positive curvature Alexandrov condition applied to the triangle $\triangle(a, p, b')$).



On this picture, $\overline{b}'' = \overline{\gamma}(t)$, and $|\overline{p}, \overline{b}'| = d(p, \gamma(t)) < |\overline{p}, \overline{\gamma}(t)| = |\overline{p}, \overline{b}''|$.

The case of non-negative curvature is considered in the same way. ■

Angles (reminder)

REMARK: In the sequel, **every time I mention $\mathbb{R}^n$, I consider it as a metric space with the Euclidean distance metric.**

DEFINITION: Let a, b, c be points in a metric space (M, d). **A comparizon triangle** $\triangle(\overline{a}, \overline{b}, \overline{c})$ is a triangle in $\mathbb{R}^2$, with vertices $\overline{a}, \overline{b}, \overline{c}$, and side lengths $|\overline{a}, \overline{b}| = d(a, b)$, $|\overline{a}, \overline{c}| = d(a, c)$, $|\overline{b}, \overline{c}| = d(b, c)$. **This triangle exists, and is uniquely determined, up to an isometry,** (the existence follows from the triangle inequality). The angle $\measuredangle(\overline{a}, \overline{b}, \overline{c}) \in [0, \pi]$ in the triangle $\overline{a}, \overline{b}, \overline{c}$ is denoted $\theta(a, b, c)$; it is called **the comparizon angle**.

DEFINITION: Let $\gamma_1 : [0, a] \longrightarrow M$, $\gamma_2 : [0, b] \longrightarrow M$ two paths in a metric space M, with $\gamma_1(0) = \gamma_2(0) = p$. **The angle** between the paths γ_1, γ_2 is the number

$$\measuredangle(\gamma_1, p, \gamma_2) := \lim_{t,s\to 0} \theta(\gamma_1(t), p, \gamma_2(s)),$$

if the limit is defined (otherwise, we say that the angle does not exist). The **upper angle** is

$$\measuredangle_{\sup}(\gamma_1, p, \gamma_2) := \limsup_{t,s\to 0} \theta(\gamma_1(t), p, \gamma_2(s)).$$

EXERCISE: Prove that **the angle between smooth paths in $\mathbb{R}^n$ exists and is equal to the angle between their tangents**.

Monotonicity of angles

Further on, when I say "monotonously increasing" (decreasing), it always means non-strict monotonicity. That is, the constant is monotonously increasing **and** monotonously decreasing.

DEFINITION: Let $\gamma_1, \gamma_2 : [0, a] \longrightarrow M$ be geodesics in M, with $\gamma_i(0) = p$. We say that M **satisfies the monotonicity for angles with non-positive/non-negative curvature** if the angle $\theta(\gamma_1(s), p, \gamma_2(t))$ monotonously increases/decreases as a function of s, t.

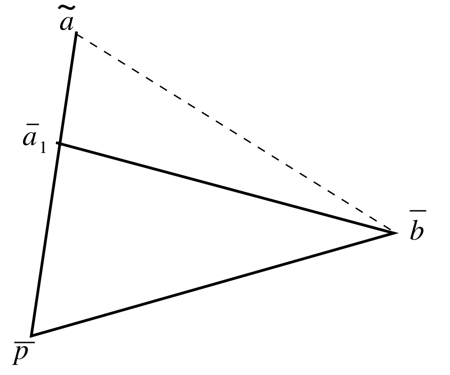
REMARK: Monotonicity of angles **immediately implies that the angle $\measuredangle(\gamma_1, p, \gamma_2) := \lim_{t,s\to 0} \theta(\gamma_1(t), p, \gamma_2(s))$ exists.** Indeed, all bounded monotonous sequences converge. Also, **when angles are monotonously decreasing,** (that is, for non-negative curvature) **the angle between distinct geodesics is always positive.**

THEOREM: Let M be a geodesic metric space. Then **the monotonicity for angles for non-positive (or non-negative) curvature is equivalent to the Alexandrov conditions $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) on M.**

Monotonicity of angles (2)

THEOREM: Let M be a geodesic metric space. Then **the monotonicity for angles for non-positive (or non-negative) curvature is equivalent to the Alexandrov conditions $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) on M.**

Proof. Step 1: Let $p, a, b \in M$, and let a_1 be a point on the minimizing geodesic connecting a to p. Consider the comparizon triangle $\triangle(\overline{a}_1, \overline{p}, \overline{b})$ for a_1, p, b and let $\tilde{a}$ be a point on the line $(\overline{p}, \overline{a}_1) \subset \mathbb{R}^2$ satisfying $|\overline{p}, \tilde{a}| = d(p, a)$.



Then $\theta(a_1, p, b) \leqslant \theta(a, p, b) \Leftrightarrow |\tilde{a}, \overline{b}| \leqslant d(a, b)$. Indeed, $|\tilde{a}, \overline{b}|$ and $d(a, b)$ are the lengths of the opposite to $\overline{p}$ sides of the triangles $\triangle(\tilde{a}, \overline{p}, \overline{b})$ and $\triangle(\overline{a}, \overline{p}, \overline{b})$, with two sides adjacent to $\overline{p}$ equal to $a(a, p)$ and $d(b, p)$. The angle at $\overline{p}$ equal to $\theta(a_1, p, b)$ for $\triangle(\tilde{a}, \overline{p}, \overline{b})$ and to $\theta(a, p, b)$ for $\triangle(\overline{a}, \overline{p}, \overline{b})$.

Step 2: The curvature conditions $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) **are equivalent to $|\tilde{a}, \overline{b}| \leqslant d(a, b)$ for non-positive curvature, and to $|\tilde{a}, \overline{b}| \geqslant d(a, b)$ for non-negative.** ■

Angles in Alexandrov spaces

COROLLARY: Let M be an Alexandrov space. **Then the angles between geodesics are always well defined.** ■

DEFINITION: Let p be an interior point on a minimizing geodesic γ. Denote two its segments, starting at p, as $\gamma_+, \gamma_- : [0,r] \longrightarrow M$. Let μ be another geodesic starting at p. **Adjacent angles** are angles $\measuredangle(\gamma_+, p, \mu)$ and $\measuredangle(\gamma_-, p, \mu)$.

REMARK: By the triangle inequality, **the sum of adjacent angles is $\geqslant \pi$.**

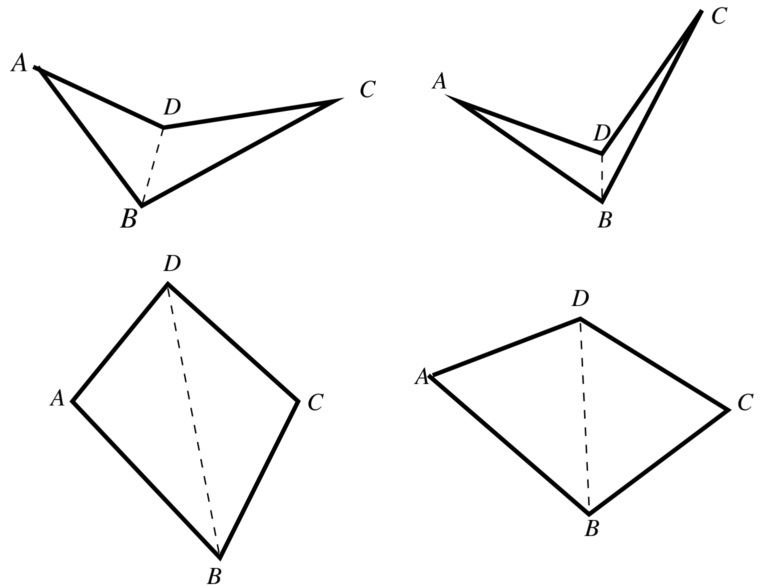
DEFINITION: Let a, b, c be three points on a metric space, and $\triangle(\overline{a}, \overline{b}, \overline{c})$ the comparison triangle. Consider the minimizing geodesics γ_1, γ_2, connecting a to b and a to c. **The angle comparison condition for non-positive curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \leqslant \measuredangle(\overline{c}\overline{a}\overline{b})$. **The angle comparison condition for non-negative curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \geqslant \measuredangle(\overline{c}\overline{a}\overline{b})$ together with the equality $\measuredangle(\gamma_+, p, \mu) + \measuredangle(\gamma_-, p, \mu) = \pi$ for any two adjacent angles $\measuredangle(\gamma_+, p, \mu)$, $\measuredangle(\gamma_-, p, \mu)$.

THEOREM: The angle comparison condition is equivalent to the curvature condition $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) for the same sign of the curvature.

Proof: Later today.

Alexandrov lemma

LEMMA: (Alexandrov lemma)
Consider a hinge mechanism on the plane consisting of four rigid rods connected by hinges. Assume that the quartilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ on the plane are obtained by deforming the mechanism (keeping the rods whole and connected). Mathematically, this means that $|A_1B_1| = |A_2B_2|$, $|B_1C_1| = |B_2C_2|$, $|C_1D_1| = |C_2D_2|$, $|D_1A_1| = |D_2A_2|$).



Then $\measuredangle(A_1B_1C_1) > \measuredangle(A_2B_2C_2) \Leftrightarrow |B_1D_1| > |B_2D_2|$, if the triangles $\triangle(A_iB_iC_i)$ and $\triangle(A_iD_iC_i)$ sit on the same side of the line (A_iC_i), and $\measuredangle(A_1B_1C_1) > \measuredangle(A_2B_2C_2) \Leftrightarrow |B_1D_1| < |B_2D_2|$ otherwise. In other words, **if we squeese the mechanism, making the angle $\measuredangle(ABC)$ smaller, the point D gets further from B for the convex quadrilateral, and gets closer for non-convex quadrilateral.**

Alexandrov lemma (2)

REMARK: Speaking of angles of the vertices of a quadrilateral, it makes sense to distinguish between **the angle** $\measuredangle(ABC)$, which is always $\leqslant \pi$, and **the plane angle**, which might be $\geqslant \pi$ if the quadrilateral is not convex in B.

Proof. Step 1: Squeezing of the hinge mechanism which decreases $|BD|$ **also decreases the angles of the quadrilateral at A and C;** this is clear from monotonicity of the third side of the angle as a function of the opposite angle.

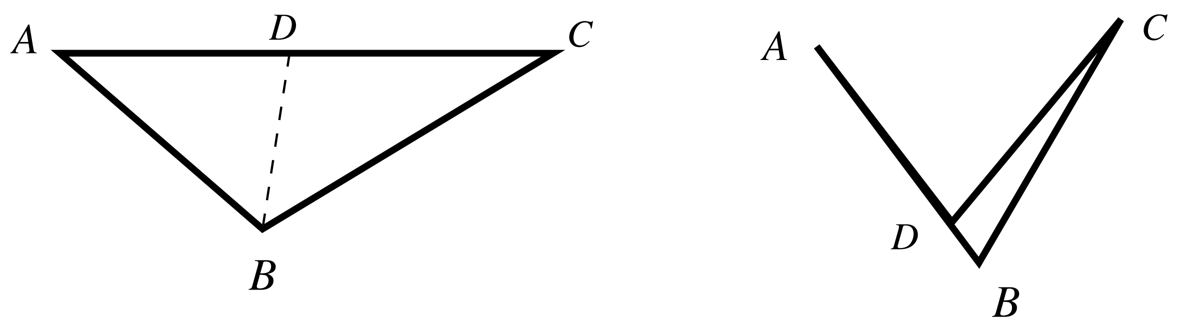
Step 2: The angles $\measuredangle(ABC)$ and $\measuredangle(ADC)$ **decrease simultaneously as $|AC|$ decreases and increase as it increases.** For a convex quadrilateral, the sum of its 4 angles is 2π. Therefore, $|BD|$ decreases if and only if $\measuredangle(BCD)$ and $\measuredangle(BAD)$ decrease, and this happens simultaneously with the increase in $|AC|$, $\measuredangle(ABC)$ and $\measuredangle(ADC)$.

Alexandrov lemma (2)

Step 3: It remains to consider the case when the plane angle $\measuredangle(ADC) \geqslant \pi$.

The following argument is due to I. Frolov.

When $\measuredangle(ADC) \geqslant \pi$, the quadrilateral $(ADCB)$ is uniquely determined if we fix the lengths of its side an either $|BD|$ or $|AC|$. This implies that the map taking the set of all possible lengths of $|BD|$ when $\measuredangle(ADC) \geqslant \pi$ to the set of all possible $|AC|$ is bijective. A bijective, continuous map from an interval to an interval is monotonous. It remains to check whether it is monotonously increasing or decreasing. This is seen from considering two limit case quadrilaterals such as below



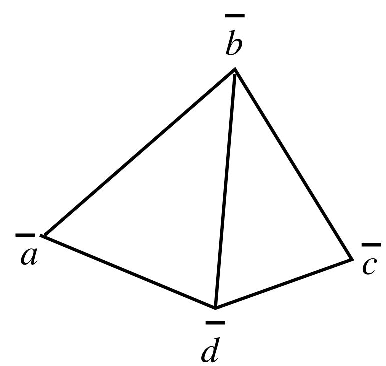
and observing that $|BD|$ in the second quadrilateral is less than in the first one, because the angle $\measuredangle(BAD)$ is smaller in the second.

■

The angle comparizon condition

THEOREM: The angle comparison condition is equivalent to the curvature condition $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) for the same sign of the curvature.

Proof. Step 1: Consider a geodesic triangle $\triangle(abc)$ in M, and let d be a point on the geodesic $[a,c] \subset M$. Place the comparison triangles $\triangle(\overline{a}\overline{bd}) \subset \mathbb{R}^2$ for $\triangle(abd) \subset M$ and $\triangle(\overline{c}\overline{bd})$ for $\triangle(cbd) \subset M$ on a plane on different sides of an interval $|\overline{bd}|$.



Convexity of this quadrilateral for all a,b,c,d means that $d_{\overline{b}} \geqslant d_b$, and non-convexity that $d_{\overline{b}} \geqslant d_b$.

Step 2: If M satisfies the angle comparison condition for non-negative curvature, and the sum of adjacent angles is π, this gives

$$\measuredangle(\overline{c},\overline{d},\overline{b}) + \measuredangle(\overline{a},\overline{d},\overline{b}) \leqslant \measuredangle(c,d,b) + \measuredangle(a,d,b) = \pi;$$

this implies that the plane angle of $(\overline{a},\overline{b},\overline{c},\overline{d})$ in $\overline{d}$ is $\leqslant \pi$, hence Alexandrov lemma, implies that $d_{\overline{b}} \geqslant d_b$. This proves the Aleksanrov comparison condition.

The angle comparizon condition (2)

Step 3: If M satisfies the angle comparison condition for non-positive curvature,

$$\measuredangle(\bar{c},\bar{b},\bar{d})+\measuredangle(\bar{a},\bar{b},\bar{d})\geqslant\measuredangle(c,b,d)+\measuredangle(a,b,d)\geqslant\pi.$$

hence by Alexandrov lemma the quadrilateral $(\bar{a},\bar{b},\bar{c},\bar{d})$ is not strictly convex (one of its plane angles is $\geqslant\pi$), which is equivalent to $d_{\bar{b}}\geqslant d_b$ (Step 1).

Step 4: To obtain the angle comparison from the Alexandrov curvature conditions, we use the monotonicity of the angles. Consider a geodesic triangle $\triangle(a_1,a_2,p)$ in M, and let $\gamma_i:[0,r_i]\longrightarrow M$ parametrize its sides adjacent to p, with $\gamma_i(0)=p$ and $\gamma_i(r_i)=a_i$. For non-positive curved Alexandrov space, the angle $\theta(\gamma_1(t),p,\gamma_2(u))$ decreases monotonously to $\measuredangle(\gamma_1,p,\gamma_2)$ as t,u decrease to 0, giving $\theta(a_1,p,a_2)\geqslant\measuredangle(\gamma_1,p,\gamma_2)$, and for non-negative curved space, it increases monotonously, giving $\theta(a_1,p,a_2)\leqslant\measuredangle(\gamma_1,p,\gamma_2)$. ■