Metric spaces

lecture 9: Convexity in CAT(0)-spaces

Misha Verbitsky

IMPA, sala 236

January 23, 2022, 17:00

Convex sets and convex functions

DEFINITION: Let M be a geodesic metric space. A subset $U \subset M$ is called **convex**, or **geodesically convex**, if for any points $x, y \in U$ any minimizing geodesic connecting x to y is contained in U, Denote by U° the interior of U. A convex set U is **strictly convex** if its closure $\overline{U}$ is convex, and for any $x, y \in \overline{U}$, the minimizing geodesic connecting x to y is contained in $U^\circ \cup \{x, y\}$.

DEFINITION: We identify a minimizing geodesic $\gamma: [a,b] \longrightarrow M$ with the interval $[a,b]$. A function on a metric space is called **convex** if its restriction to any minimizing geodesic is convex, and **strictly convex** if its restriction to any minimizing geodesic is strictly convex, that is, is not linear on any smaller interval $[x,y] \subset [a,b]$.

EXERCISE: Let $f: M \longrightarrow \mathbb{R}$ be a (strictly) convex function. **Prove that the set $f^{-1}(]-\infty, c])$ is convex (strictly convex).**

Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points on a geodesic metric space (M, d), and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An intrinsic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for all geodesic triangles.

REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

Convex functions in CAT(0)-spaces

PROPOSITION: Let $z \in M$, where M is a CAT(0)-space. **Then the function $d_z(x) := d(z,x)$ is convex,** and **strictly convex on any geodesic segment connecting a to b if the triangle inequality for z, a, b is strict.**

Proof: Let $\gamma: [0,t] \longrightarrow M$ be a minimising geodesic connecting a to b, and $\triangle(\overline{a}, \overline{b}, \overline{z})$ the comparison triangle. Then $d_z \leqslant d_{\overline{z}}$, but the last function is (strictly) convex, giving

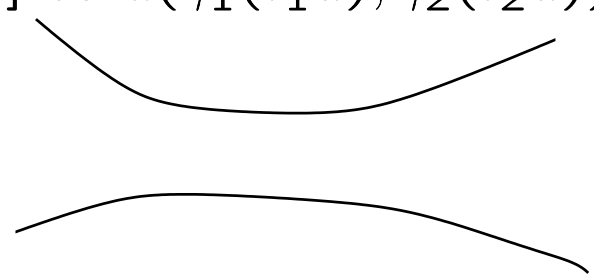
$$d_z(\lambda t) \leqslant d_{\overline{z}}(\lambda t) < \lambda d_{\overline{z}}(0) + (1-\lambda) d_{\overline{z}}(t) = \lambda d_z(0) + (1-\lambda) d_z(t). \quad \blacksquare$$

PROPOSITION: Let $y, z \in M$ be points in a CAT(0)-soace. **Then a minimizing geodesic connecting z to y is unique.**

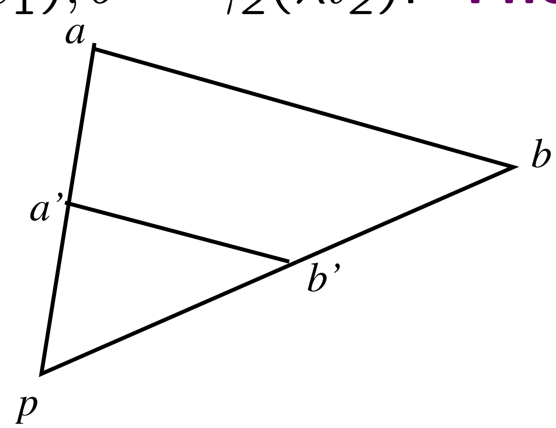
Proof: Let A, B be geodesics connecting z to y, and a a point on A. The triangle $\triangle(ayz)$ is degenerate, because $|A| = d(y,z)$ and $\frac{1}{2}|A| = d(y,a) = d(z,a)$. Let $\triangle(\overline{ayz})$ be the comparison triangle. Since $\triangle(\overline{ayz})$ degenerate, the distance between $\overline{a}$ and a certain point on the opposite side is zero; CAT(0)-inequalities imply that $d(a,b) = 0$, for the corresponding point b on B. $\blacksquare$

Distance between geodesics in CAT(0)-spaces

LEMMA: (Convexity lemma)
Let $\gamma_i : [0, t_i] \longrightarrow M$, $i = 1, 2$ be geodesics in a CAT(0)-space, and $\kappa : [0, 1] \longrightarrow \mathbb{R}^{\geqslant 0}$ takes $u \in [0, 1]$ to $d(\gamma_1(t_1 u), \gamma_2(t_2 u))$. **Then κ is convex.**



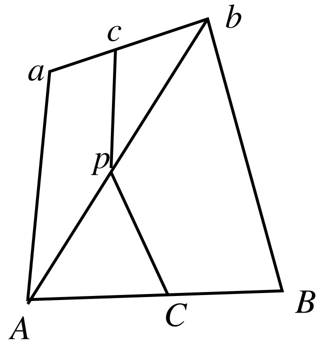
Proof. Step 1: Let $\gamma_1(t_1) = a, \gamma_2(t_2) = b$ be points on the geodesics. Choose $0 < \lambda < 1$, and let $a' = \gamma_1(\lambda t_1), b' = \gamma_2(\lambda t_2)$. **Then $d(a', b') \leqslant \lambda d(a, b)$,**



by the angle monotonicity property.

Distance between geodesics in CAT(0)-spaces (2)

Step 2: Let a, b, A, B be points in a CAT(0)-space, and c, C the middle points of the geodesics (a, b) and (A, B). Denote by p the middle point of the geodesic (A, b)



Step 1 implies that

$$d(c,p) + d(p,C) \leqslant \frac{1}{2}(d(a,A) + d(b,B)),$$

and the triangle inequality implies $d(c,C) \leqslant d(c,p) + d(p,C)$. **This implies $\kappa(1/2) \leqslant \frac{1}{2}(\kappa(0) + \kappa(1))$.**

Step 3: We obtained that for any subinterval $[a,b] \subset [0,1]$, one has

$$\kappa\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}(\kappa(a) + \kappa(b)).$$

Step 4: For any continuous function $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$, **the inequality $\kappa(\frac{a+b}{2}) \leqslant \frac{1}{2}(\kappa(a) + \kappa(b))$ implies convexity (check this).** ■

Uniform distance on the space of geodesics

DEFINITION: Define **the uniform distance** between continuous maps $\gamma_i : [0, t_i] \longrightarrow M$ as $d_\Gamma(\gamma_1, \gamma_2) := \sup_{x\in[0,1]} d(\gamma_1(x/t_1), \gamma_2(x/t_2))$. The corresponding convergence of sequences is called **the uniform convergence**.

EXERCISE: Check that it is a metric.

THEOREM: Let M be a CAT(0)-space, and $(\Gamma_p(M), d_\Gamma)$ be the metric space of all normally parametrized geodesics $\gamma : [0, t] \longrightarrow M$, $\gamma(0) = p$, $t \in \mathbb{R}$. Let $\pi : \Gamma_p(M) \longrightarrow M$ take the geodesic $\gamma : [0, t] \longrightarrow M$ to $\gamma(t)$. **Then π is an isometry.**

Proof: Let γ, γ' be geodesics in a CAT(0)-space, and a, b and a', b' their ends. Then $d_\Gamma(\gamma, \gamma') = \sup_{t\in[0,1]}(\kappa(t)) = \max(d(a, a'), d(b, b'))$, **because κ is convex, and the convex function on an interval reaches its maximum on its boundary.** ■

This argument also proves

THEOREM: Let M be a CAT(0)-space, and $\gamma_i : [0, t_i] \longrightarrow M$ a sequence of (normal parametrized) geodesics such that the ends $a_i := \gamma_i(0)$, $b_i := \gamma_i(t_i)$ converge to a, b. Then **the sequence $\gamma_i : [0, t_i] \longrightarrow M$ uniformly converges to the geodesic $\gamma : [0, t] \longrightarrow M$ connecting a to b.** ■