Metric spaces

lecture 10: Cartan-Hadamard theorem

Misha Verbitsky

IMPA, sala 236

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Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points on a geodesic metric space (M, d), and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An intrinsic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for all geodesic triangles.

REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

PROPOSITION: Let $z \in M$, where M is a CAT(0)-space. **Then the function $d_z(x) := d(z, x)$ is convex**.

Distance between geodesics in CAT(0)-spaces (reminder)

LEMMA: (Convexity lemma)
Let $\gamma_i : [0,t_i] \longrightarrow M$, $i = 1,2$ be geodesics in a CAT(0)-space, and $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ takes $u \in [0,1]$ to $d(\gamma_1(t_1u), \gamma_2(t_2u))$. **Then κ is convex.**

DEFINITION: Define **the uniform distance** between continuous maps $\gamma_i : [0,t_i] \longrightarrow M$ as $d_\Gamma(\gamma_1,\gamma_2) := \sup_{x\in[0,1]} d\left(\gamma_1(xt_1), \gamma_2(xt_2)\right)$. The corresponding convergence of sequences is called **the uniform convergence**.

EXERCISE: Check that it is a metric.

THEOREM: Let M be a CAT(0)-space, and $(\Gamma_p(M), d_\Gamma)$ be the metric space of all normally parametrized geodesics $\gamma : [0,t] \longrightarrow M$, $\gamma(0) = p$, $t \in \mathbb{R}$. Let $\pi : \Gamma_p(M) \longrightarrow M$ take the geodesic $\gamma : [0,t] \longrightarrow M$ to $\gamma(t)$. **Then π is an isometry.**

REMARK: Let M be a CAT(0)-space, and $\gamma_i : [0,t_i] \longrightarrow M$ a sequence of (normal parametrized) geodesics such that the ends $a_i := \gamma_i(0)$, $b_i := \gamma_i(t_i)$ converge to a, b. Then **the sequence $\gamma_i : [0,t_i] \longrightarrow M$ uniformly converges to the geodesic $\gamma : [0,t] \longrightarrow M$ connecting a to b.**

Homotopy and the space of geodesics

Fix a point p in a CAT(0)-space M. Let $0 \leqslant \lambda \leqslant 1$, and let $P_\lambda : \Gamma_p(M) \longrightarrow \Gamma_p(M)$ take the geodesic $\gamma : [0,t] \longrightarrow M$ to $\gamma\Big|_{[0,\lambda t]}$.

REMARK: Since $\Gamma_p(M)$ is isometric to M, one can consider P_λ as a map from M to itself. By convexity lemma, $d(\gamma(\lambda t), \gamma_1(\lambda t_1)) \leqslant d(\gamma(t), \gamma_1(t_1))$, hence **the map P_λ is 1-Lipschitz.**

REMARK: Let $P : M \times [0,1] \longrightarrow M$ take (m,t) to $P_t(m)$. **Since P_λ is 1-Lipschitz, P is also 1-Lipschitz, hence continuous.**

COROLLARY: Any CAT(0)-space is contractible.

Proof: The map P defines a homotopy between identity and the map taking M to p. ■

The convexity radius

DEFINITION: Let M be an Alexandrov space of non-positive curvature. A **normal ball** in M is a ball $B_x(\varepsilon)$ which is a CAT(0)-space.

REMARK: **An open ball in a CAT(0)-space is normal.** Indeed, **CAT(0)-inequalities imply that it is convex.**

DEFINITION: Let M be an Alexandrov space of non-positive curvature. **Convexity radius** of M in $x \in M$ is the supremum of all ε such that $B_x(\varepsilon)$ is a normal ball.

CLAIM: Denote the convexity radius in x by $\rho(x)$. **Then the function ρ is 1-Lipschitz.**

Proof: Follows from the "standard argument" (we have used it a few times already).

General form of the "standard argument": Let $\mathfrak{S}$ be a subset in the set of all balls in a metric space, such that for any ball $B_x(r) \in \mathfrak{S}$, all ball contained in $B_x(r)$ also belong to $\mathfrak{S}$. **Then the function $\rho_{\mathfrak{S}}(x) := \sup_r \{r \mid B_r(x) \in \mathfrak{S}\}$ is 1-Lipschitz.**

EXERCISE: Prove this statement.

Convexity radius for a subset

DEFINITION: **Convexity radius** for a subset $Z \subset M$ is $\inf_{z\in Z} \rho(z)$, where $\rho(z)$ denotes the convexity radius of M in $z \in Z$.

CLAIM: **Convexity radius is positive** for any compact subset $Z \subset M$ in an Alexandrov space M of non-positive curvature.

Proof: Small open balls in M are convex, hence $\rho(z) > 0$ for all $z \in M$. Since ρ is Lipschitz, it is continuous, then **ρ reaches its minimum (which is positive) on any compact $Z \subset M$.**

Convexity lemma for non-minimizing geodesics

DEFINITION: A geodesic is a map $\gamma : [a,b] \longrightarrow M$ such that $[a,b]$ is a union of intervals $[a,b] = \bigcup_i [x_i, x_{i+1}]$, and $\gamma\Big|_{[x_i,x_{i+1}]}$ is a minimizing geodesic. Denote by $\Gamma(M)$ the space of all normally parametrized geodesics with the metric d_Γ, and let $\Gamma_p(M) \subset \Gamma(M)$ denote the space of geodesics starting in p.

PROPOSITION: Let $\gamma : [0,t] \longrightarrow M$, $\gamma' : [0,t'] \longrightarrow M$ be geodesics in an Alexandrov space of non-positive curvature. Assume that the convexity radius in γ is equal to ε, and $d_\Gamma(\gamma,\gamma') < \frac{1}{2}\varepsilon$. Define $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ as $\kappa(u) := d(\gamma(ut), \gamma'(ut'))$. **Then κ is a convex function.**

Proof: Convexity of a function $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ is local in its domain. The geodesics γ and γ' can be obtained as a union of the minimizing fragments $\gamma\Big|_{[\lambda t, (\lambda + t^{-1}\delta)t]}$ and $\gamma'\Big|_{[\lambda t', (\lambda + t^{-1}\delta)t']}$ of length $\leqslant \delta$ which sit in normal balls if $\delta < \frac{\varepsilon \min t, t'}{2t}$, hence **the the restriction of κ to each interval $[\lambda, \lambda + t^{-1}\delta]$ is convex.** ■

Distance between the non-minimizing geodesics

COROLLARY: Let $\gamma : [0,t] \longrightarrow M$, $\gamma' : [0,t'] \longrightarrow M$ be normally parametrized geodesics in an Alexandrov space of non-positive curvature, radius of convexity of γ is ε, and $d_\Gamma(\gamma,\gamma') < \varepsilon/2$. **Then the distance between geodesics is the maximum of the distance between their ends,**

$$d_\Gamma(\gamma,\gamma') = \max(d(\gamma(0),\gamma'(0)), d(\gamma(t),\gamma'(t')))$$

Proof: The convex function $\kappa : [0,1] \longrightarrow \mathbb{R}$ takes its maximum in 0 or 1. ■

COROLLARY: Let $\Gamma_p(M) \xrightarrow{\pi} M$ take a geodesic to its second end. Let ε be the convexity radius for γ. **Then π defines an isometry between $B_{1/2\varepsilon}(\gamma) \subset \Gamma_p(M)$ and $B_{1/2\varepsilon}(\pi(\gamma))$.** ■

Covering maps

DEFINITION: Let $\varphi : \tilde{M} \longrightarrow M$ be a continuous map of manifolds (or CW complexes). We say that φ is **a covering** if φ is locally a homeomorphism, and for any $x \in M$ there exists a neighbourhood $U \ni x$ such that is a disconnected union of several manifolds U_i such that the restriction $\varphi\big|_{U_i}$ is a homeomorphism.

REMARK: From now on, **M is connected, locally contractible topological space.**

EXERCISE: Prove that **a local homeomorphism of compacts spaces is a covering.**

DEFINITION: Let Γ be a discrete group continuously acting on a topological space M. This action is called **properly discontinuous** if M is locally compact, and the space of orbits of Γ is Hausdorff.

THEOREM: Let Γ be a discrete group acting on M properly discontinuously. Suppose that the stabilizer group $\Gamma' : \mathrm{St}_\Gamma(x)$ is the same for all $x \in M$. **Then $M \longrightarrow M/\Gamma$ is a covering.** Moreover, **all covering maps are obtained like that.**

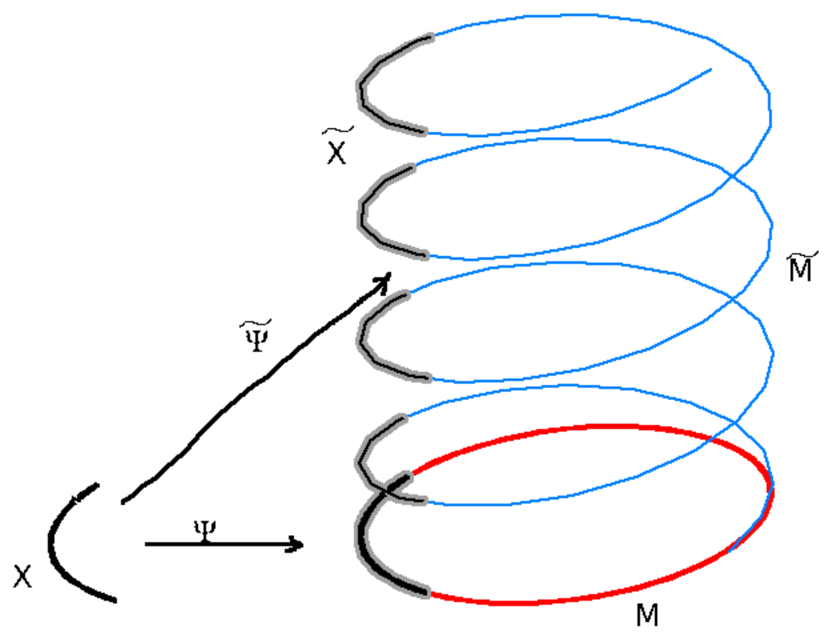
Proof: Left as an exercise.

Homotopy lifting

DEFINITION: A topological space M is **locally contractible** if any point of M has a contractible neighbourhood.

REMARK: Non-positive curvature Alexandrov spaces **are locally CAT(0), hence locally contractible.**

LEMMA: ("Homotopy lifting lemma")
Let M be a locally contractible topological space. The map $\varphi: \tilde M \longrightarrow M$ is a covering iff φ is locally a homeomorphism, and for any path $\Psi: [0,1] \longrightarrow M$ and any $x \in \varphi^{-1}(\Psi(0))$, **there is a lifting $\tilde\Psi: [0,1] \longrightarrow \tilde M$ such that $\tilde\Psi(0) = x$ and $\varphi(\tilde\Psi(t)) = \Psi(t)$.** Moreover, **the lifting is uniquely determined by the homotopy class of Ψ in the set of all paths connecting $\Psi(0)$ to $\Psi(1)$.**



Homotopy lifting

COROLLARY: If M is simply connected, **all connected coverings $\tilde{M} \longrightarrow M$ are isomorphic to M.** ■

REMARK: If M is a geodesic metric space, **to prove that $\varphi: \tilde{M} \longrightarrow M$ is a covering, it is sufficient to prove homotopy lifting for piecewise geodesic paths**. **This would follow if we prove the homotopy lifting for geodesics.**

Local convexity

DEFINITION: Let $U \subset M$ be an open subset in a negatively curved locally compact Alexandrov space. We say that U is **strictly convex** if any two points in $\overline{U}$ can be connected by a geodesic which belongs to $\overline{U}$, and any geodesic connecting x to $y \in \overline{U}$ belongs to $U \cup \{x, y\}$.

EXAMPLE: Let ε be the convexity radius of a geodesic $\gamma \subset M$. **Then $\varepsilon/2$-neighbourhood of any $\varepsilon/2$-segment of γ is strictly convex.**

PROPOSITION: Let $\gamma \subset M$ be a finite geodesic in a negatively curved locally compact Alexandrov space (M, d), ε its convexity radius, and U an $\varepsilon/2$-neighbourhood of γ. **Then U is strictly convex.**

Proof. Step 1: Let $\tilde{K}$ be the universal cover of $K := \overline{U}$, equipped with a local metric $\tilde{d}$ induced from M. The metric $\tilde{d}$ on $\tilde{K}$ is geodesic by Cohn-Vossen theorem: indeed, it is local, hence intrinsic, and $\tilde{K}$ is locally compact. Also, $d_K \geqslant d$. Denote by $\Gamma(\tilde{K})$ the space of finite length geodesics in $\tilde{K}$ equipped with a uniform metric induced by $\tilde{d}$. As we have already established, the natural map $\Gamma(\tilde{K}) \longrightarrow \tilde{K} \times \tilde{K}$ is locally an isometry; since the limit of the geodesic is a geodesic, its image is complete, hence closed. Since d_K is locally equal to d, any geodesic in $(\tilde{K}, \tilde{d})$ is a geodesic in M.

Step 2: Let $\Gamma_0 \subset \Gamma(\tilde{K})$ be the set of all geodesics in $\Gamma(\tilde{K})$ connecting x to y which belong to the interior $\tilde{K}^\circ \cup \{x, y\}$. Any geodesic in $\tilde{K}$ belongs to Γ_0 or has an interior point in $\partial\tilde{K}$. The later case is impossible, because in any $\varepsilon/2$-neighbourhood of an $\varepsilon/2$-segment γ_0 of γ, the uniform distance function κ associated with γ_1 and γ_0 is strictly convex, hence it cannot reach the maximum $\varepsilon/2$ in any interior point of γ_1. **Therefore, $\Gamma_0 = \Gamma(\tilde{K})$.** ■

COROLLARY: In these assumptions, **the natural map $\Pi: \Gamma(\tilde{K}) \longrightarrow \tilde{K} \times \tilde{K}$, taking a geodesic to its ends, is bijective and preserves the distance.**

Proof: From convexity of the function κ, we obtain that this map preserves the metric; it is surjective, because $\tilde{K}$ is a geodesic space, and injective, because the κ function associated with two geodesics with the same ends in an $\varepsilon/2$-neighbourhood of γ is convex, hence it vanishes identically. ■

Cartan-Hadamard theorem

DEFINITION: A complete, simply connected Alexandrov space of non-positive curvature is called **a Hadamard space**.

THEOREM: (Cartan-Hadamard)
Let M be a Hadamard space. Consider the map $\Gamma_p(M) \xrightarrow{\pi} M$ taking a geodesic to its second end. **Then π is a homeomorphism.**
Proof: Later today.

COROLLARY: Any geodesic in a Hadamard space M is minimizing. Moreover, **any two points in M can be connected by a unique geodesic.**

COROLLARY: Any Hadamard space is contractible.

Proof: The homotopy $P_\lambda : \Gamma_p(M) \longrightarrow \Gamma_p(M)$ which defines a contraction of M to a point is 1-Lipschitz in any ball of radius less than the convexity radius, hence continuous. Therefore, it defines a comotopy between the identity map and a projection to a point. ■

The proof of Cartan-Hadamard theorem

THEOREM: (Cartan-Hadamard)
Let M be a complete Alexandrov space of non-positive curvature. Consider the map $\Gamma_p(M) \xrightarrow{\pi} M$ taking a geodesic to its second end. **Then π is a covering.**

Proof. Step 1: Let $\pi : X \longrightarrow Y$ be a local isometry of complete metric spaces with geodesic metric. As explained today, homotopy lifting for piecewise geodesics implies that π is a covering. I am going to prove that **such a map is always a covering, if X is a complete metric space.**

Step 2: Let $\gamma : [0, r] \longrightarrow Y$ be a piecewise geodesic path in Y. Locally in a neighbourhood of any point the homotopy lifting holds, because π is a local isometry; also, a lifting $\tilde{\gamma}$ of γ to X on an interval $[0, a]$ is uniquely determined by the point $\tilde{\gamma}(0) \in X$. Fix $x \in \pi^{-1}(\gamma(0))$, and let $r_0 \in [0, r]$ be the supremum of all points $t \in [0, r]$ such that the lifting $\tilde{\gamma} : [0, t] \longrightarrow X$, starting from $\tilde{\gamma}(0) = x$, exists. Put $\tilde{\gamma}(r_0) := \lim_{t \to r} \tilde{\gamma}(t)$. This limit exists and is well defined, because $\tilde{\gamma}$ is a local isometry, and X is complete. Since π is a local isometry in a neighbourhood of $\tilde{\gamma}(r_0)$, **the lifting γ can be extended to some neighbourhood of $r_0 \in [0, r]$, which implies that $r_0 = r$.** ■