Metric spaces

lecture 11: Hadamard spaces are CAT(0)

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Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points on a geodesic metric space (M, d), and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An intrinsic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for all geodesic triangles.

REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

PROPOSITION: Let $z \in M$, where M is a CAT(0)-space. **Then the function $d_z(x) := d(z, x)$ is convex.**

Angles in Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points in a metric space (M, d). **A comparizon triangle** $\triangle(\overline{a}, \overline{b}, \overline{c})$ is a triangle in $\mathbb{R}^2$, with vertices $\overline{a}, \overline{b}, \overline{c}$, and side lengths $|\overline{a}, \overline{b}| = d(a, b)$, $|\overline{a}, \overline{c}| = d(a, c)$, $|\overline{b}, \overline{c}| = d(b, c)$. **This triangle exists, and is uniquely determined, up to an isometry,** (the existence follows from the triangle inequality). The angle $\measuredangle(\overline{a}, \overline{b}, \overline{c}) \in [0, \pi]$ in the triangle $\overline{a}, \overline{b}, \overline{c}$ is denoted $\theta(a, b, c)$; it is called **the comparizon angle**.

DEFINITION: Let a, b, c be three points on a metric space, and $\triangle(\overline{a}, \overline{b}, \overline{c})$ the comparison triangle. Consider the minimizing geodesics γ_1, γ_2, connecting a to b and a to c. **The angle comparison condition for non-positive curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \leqslant \measuredangle(\overline{c}\overline{a}\overline{b})$. **The angle comparison condition for non-negative curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \geqslant \measuredangle(\overline{c}\overline{a}\overline{b})$ together with the equality $\measuredangle(\gamma_+, p, \mu) + \measuredangle(\gamma_-, p, \mu) = \pi$ for any two adjacent angles $\measuredangle(\gamma_+, p, \mu)$, $\measuredangle(\gamma_-, p, \mu)$.

THEOREM: The angle comparison condition is equivalent to the Alexandrov condition $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) for the same sign of the curvature.

Distance between geodesics in CAT(0)-spaces (reminder)

LEMMA: (Convexity lemma)
Let $\gamma_i : [0,t_i] \longrightarrow M$, $i = 1,2$ be geodesics in a CAT(0)-space, and $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ takes $u \in [0,1]$ to $d(\gamma_1(t_1 u), \gamma_2(t_2 u))$. **Then κ is convex.**

DEFINITION: Define **the uniform distance** between continuous maps $\gamma_i : [0,t_i] \longrightarrow M$ as $d_\Gamma(\gamma_1,\gamma_2) := \sup_{x\in[0,1]} d\left(\gamma_1(xt_1), \gamma_2(xt_2)\right)$. The corresponding convergence of sequences is called **the uniform convergence**.

EXERCISE: Check that it is a metric.

THEOREM: Let M be a CAT(0)-space, and $(\Gamma_p(M), d_\Gamma)$ be the metric space of all normally parametrized geodesics $\gamma : [0,t] \longrightarrow M$, $\gamma(0) = p$, $t \in \mathbb{R}$. Let $\pi : \Gamma_p(M) \longrightarrow M$ take the geodesic $\gamma : [0,t] \longrightarrow M$ to $\gamma(t)$. **Then π is an isometry.**

REMARK: Let M be a CAT(0)-space, and $\gamma_i : [0,t_i] \longrightarrow M$ a sequence of (normal parametrized) geodesics such that the ends $a_i := \gamma_i(0)$, $b_i := \gamma_i(t_i)$ converge to a, b. Then **the sequence $\gamma_i : [0,t_i] \longrightarrow M$ uniformly converges to the geodesic $\gamma : [0,t] \longrightarrow M$ connecting a to b.**

The convexity radius (reminder)

DEFINITION: Let M be an Alexandrov space of non-positive curvature. A **normal ball** in M is a ball $B_x(\varepsilon)$ which is a CAT(0)-space.

DEFINITION: Let M be an Alexandrov space of non-positive curvature. **Convexity radius** of M in $x \in M$ is the supremum of all ε such that $B_x(\varepsilon)$ is a normal ball.

CLAIM: Denote the convexity radius in x by $\rho(x)$. **Then the function ρ is 1-Lipschitz.**

DEFINITION: **Convexity radius** for a subset $Z \subset M$ is $\inf_{z\in Z} \rho(z)$, where $\rho(z)$ denotes the convexity radius of M in $z \in Z$.

CLAIM: **Convexity radius is positive** for any compact subset $Z \subset M$ in an Alexandrov space M of non-positive curvature.

PROPOSITION: Let $\gamma : [0,t] \longrightarrow M$, $\gamma' : [0,t'] \longrightarrow M$ be geodesics in an Alexandrov space of non-positive curvature. Assume that the convexity radius in γ is equal to ε, and $d_\Gamma(\gamma,\gamma') < \frac{1}{2}\varepsilon$. Define $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ as $\kappa(u) := d(\gamma(ut), \gamma'(ut'))$. **Then κ is a convex function.**

Cartan-Hadamard theorem (reminder)

DEFINITION: A complete, simply connected Alexandrov space of non-positive curvature is called **a Hadamard space**.

THEOREM: (Cartan-Hadamard)
Let M be a Hadamard space. Consider the map $\Gamma_p(M) \stackrel{\pi}{\longrightarrow} M$ taking a geodesic to its second end. **Then π is a homeomorphism.**

THEOREM: (Cartan-Hadamard)
Let M be a complete Alexandrov space of non-positive curvature. Consider the map $\Gamma_p(M) \stackrel{\pi}{\longrightarrow} M$ taking a geodesic to its second end. **Then π is a covering.**

Convexity for a neighbourhood of a geodesic

LEMMA: Let M be a Hadamard space and $a, b \in M$. Then **there exists a neighbourhood $U_a \ni a, U_b \ni b$, such that for all $a' \in U_a, b' \in U_b$, the function $\kappa : [0,1] \longrightarrow \mathbb{R}^{\geqslant 0}$ as $\kappa(u) := d(\gamma(ut), \gamma'(ut'))$ associated with the geodesics $[a,b]$ and $[a',b']$ is convex.**

Proof. Step 1: Let $\pi_a : \Gamma_a(M) \longrightarrow M$ be the standard homeomorphism, and ε the convexity radius for the geodesic $[a,b]$. Then $d_\Gamma([a,b],[a,b']) = d(b,b')$ when $d_\Gamma([a,b],[a,b']) \leqslant \varepsilon/2$, **hence $d_\Gamma([a,b],[a,b']) = d(b,b')$ for all $b' \in \pi_a^{-1}(B_{[a,b]}(\varepsilon/2))$.**

Step 2: The same argument proves that for all $a' \in \pi_{b'}^{-1}(B_{[a,b']}(\varepsilon/2))$, we have $d_\Gamma([a',b'],[a,b']) = d(a,a')$. Triangle inequality implies

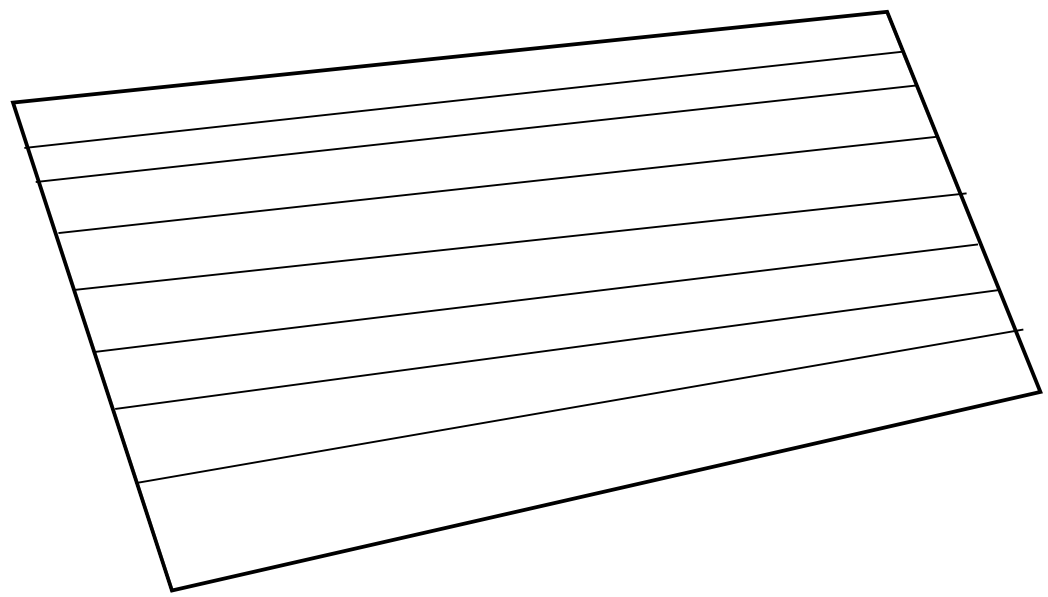
$$d_\Gamma([a,b],[a',b']) \leqslant d_\Gamma([a',b'],[a,b']) + d_\Gamma([a,b],[a,b']) \leqslant d(a,a') + d(b,b').$$

When this number is less than $\varepsilon/2$, κ is convex (Lecture 10). ■

Thin triangles and quadrilaterals

DEFINITION: A triangle is $\triangle(pxy)$ called **thin** if the function κ associated with the geodesics $[px]$ and $[py]$ is convex. A quadrilateral $(abcd)$ is called **thin** if the function κ associated with the geodesics $[ab]$ and $[dc]$ is convex.

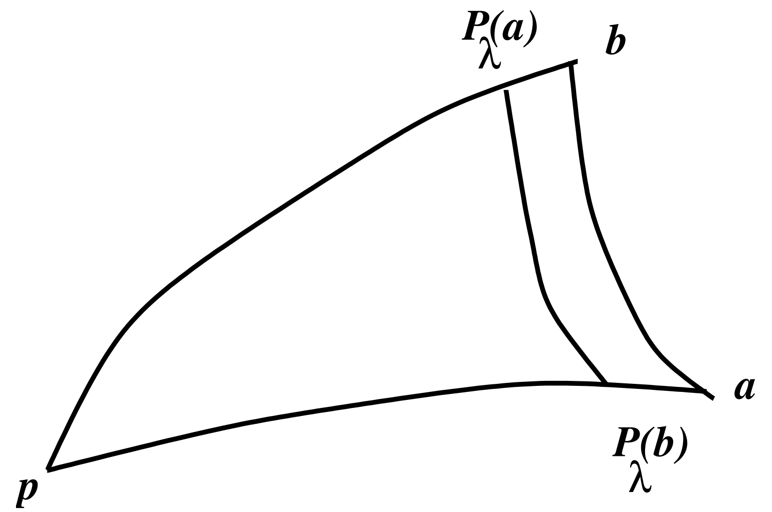
COROLLARY: Let M be a Hadamard space, and $\triangle(abcd)$ a geodesic quadrilateral. Then **it is possible to partition two opposite sides into smaller intervals and connect them by geodesics** in such a way that all successive quadrilaterals are thin.



Hadamard spaces are CAT(0)

THEOREM: Let M be a Hadamard space. **Then M is CAT(0).**

Proof. Step 1: From Cartan-Hadamard theorem it follows that a geodesic connecting two points in M is unique, moreover, the homotopy $P_\lambda : \Gamma_p(M) \longrightarrow \Gamma_p(M)$ taking a geodesic $\gamma : [0,t] \longrightarrow M$ to $\gamma\Big|_{[0,\lambda t]}$ is continuous. Consider a geodesic triangle $\triangle pab$, and let $a_\lambda = P_\lambda(a), b_\lambda = P_\lambda(b)$ be the points obtained from a, b using the identification $M = \Gamma_p(M)$.



From the angle comparison criterion (Lecture 8) we obtain that CAT(0)-inequality would follow if $\theta(apb) \leqslant \measuredangle([pa],p,[pb])$. Since

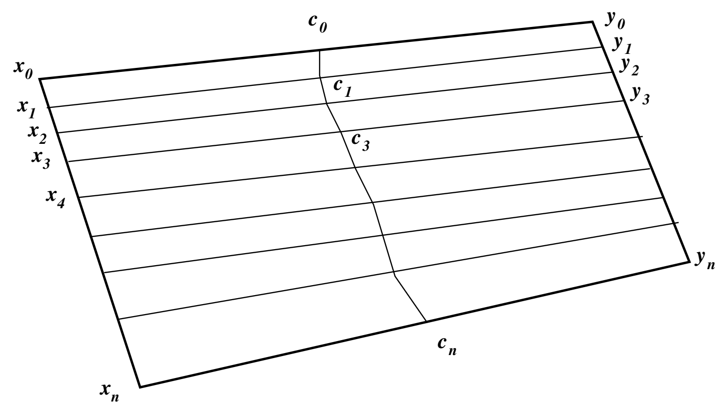
$$\measuredangle([pa],p,[pb]) = \lim_{\lambda\to o} \theta(a_\lambda p b_\lambda),$$

this **would follow if we prove that $\theta(a_\lambda p b_\lambda) \leqslant \theta(apb)$ is monotonously non-increasing as a function of λ.**

Hadamard spaces are CAT(0) (Step 2-3)

Step 2: Suppose that the function $d(a_\lambda, b\lambda)$ is convex as a function of λ. Then $\frac{1}{\lambda}d(a_\lambda, b\lambda) \leqslant d(a,b)$, hence the comparison triangle $\triangle(p, a_\lambda, b\lambda)$ scaled with λ^{-1} has two sides $d(p,a)$ and $d(p,b)$ and the third side $\frac{1}{\lambda}d(a_\lambda, b\lambda) < d(a,b)$. We obtain that $\theta(a_\lambda p b_\lambda) \leqslant \theta(apb)$ **whenever the function $\kappa(\lambda) = d(a_\lambda, b\lambda)$ is convex as a function of λ.** This function is the standard κ (the uniform distance) associated with the geodesics $[p,a]$ and $[p,b]$, hence **it is convex for thin triangles**.

Step 3: Consider a quadrilateral which is cut unto a union of thin quadrilaterals $(x_i y_i y_{i+1} x_{i+1})$ as on the picture.



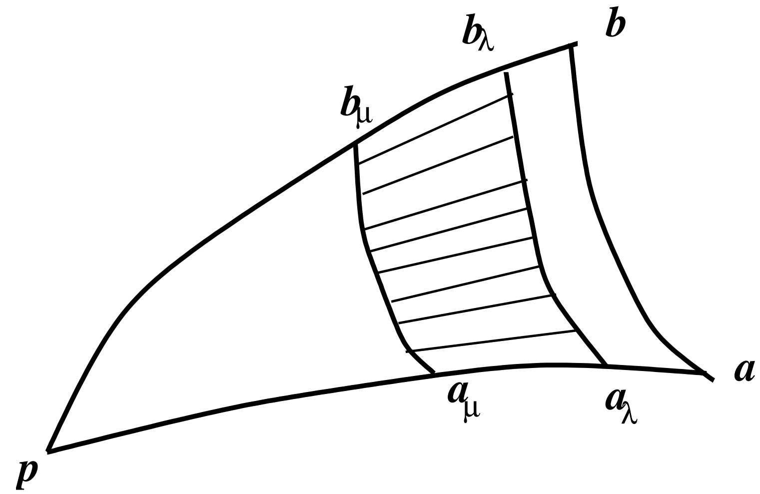
Let c_i be the middle point of each geodesic $[x_i, y_i]$. **Then**

$$d(c_0, c_n) \leqslant \sum_{i=0}^{n-1} d(c_i, c_{i+1}) \leqslant \frac{1}{2}\sum_{i=0}^{n-1} d(x_i, x_{i+1}) + d(y_i, y_{i+1}) = d(x_0, x_n) + d(y_0, y_n).$$

because each quadrilateral $(x_i y_i y_{i+1} x_{i+1})$ is thin.

Hadamard spaces are CAT(0) (Step 4)

Step 4: Consider now a triangle $\triangle(pab)$ a quadrilateral $(a_\lambda a_\mu b_\mu b_\lambda)$ cut into thin triangles as on this picture



Step 3 implies that the distance between middle points of the geodesics $[a_\lambda, a_\mu]$ and $[b_\lambda, b_\mu]$ is less than $\frac{1}{2}(d(a_\lambda, b_\lambda) + d(a_\mu, b_\mu))$. **We obtained that the function $\kappa(\lambda) := d(a_\lambda, b_\lambda)$ satisfies**

$$\kappa\left(\frac{\lambda+\mu}{2}\right) \leqslant \frac{1}{2}(\kappa(\lambda)+\kappa(\mu))$$

which implies that κ is convex. From Step 2 it follows that M is CAT(0).

■

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \ \mid \ g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on $T_x M$.

DEFINITION: A Riemannian form Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant scalar product on $T_x M$. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on $T_y M$ obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ_x is h-invariant, **the metric Φ_y is independent from the choice of g.**

We proved

THEOREM: Homogeneous Riemannian forms on $M = G/H$ are in bijective correspondence with isotropy invariant spalar products on $T_x M$, for any $x \in M$. ■

Space forms

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/O(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: **From now on, all space forms are assumed to be homogeneous Riemannian manifolds**.