Metric spaces

lecture 12: Hyperbolic space

Misha Verbitsky

IMPA, sala 236

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Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has** $M = G/H$, where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A Riemannian form Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant scalar product on T_xM. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ_x is h-invariant, **the metric Φ_y is independent from the choice of g.**

We proved

THEOREM: Homogeneous Riemannian forms on $M = G/H$ are in bijective correspondence with isotropy invariant spalar products on T_xM, for any $x \in M$. ■

Space forms

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/O(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Geodesic on space forms

LEMMA: Let $m \in M$ be a point on an n-dimensional space form (M, g). Then there exists $\varepsilon > 0$ such that for any $x \in B_m(\varepsilon)$, **the geodesic connecting m and x inside $B_m(\varepsilon)$ is unique.**

REMARK: This statement **is actually true for any Riemannian manifold,** but the proof is not elementary.

Proof. Step 1: Clearly, it suffices to prove this only for a sphere or $\mathbb{H}^n$, for $\mathbb{R}^n$ the statement is clear. We realize the space form as a sphere (for positive curvature), or as a hyperboloid

$$\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \ \mid \ x_0^2 = 1 + \sum x_i^2\}$$

for the negative curvature. The projection to the tangent plane gives flat coordinates on M in a neighbourhood of m. The stabilizer of M in the isometry group of M is $O(n)$, and any $O(n)$-invariant set in T_mM is a union of spheres. Since the image of a geodesic ball is connected, **it is a round ball in T_mM.** Therefore, any open ball in M which projects to T_mM bijectively is given in the above coordinates by an equation $\{x \in M \ \mid \ g(x, m) > \delta\}$ for $\mathbb{H}^n$ and $\{x \in M \ \mid \ g(x, m) < \delta\}$ for S^n.

Geodesic on space forms

LEMMA: Let $m \in M$ be a point on an n-dimensional space form (M, g). Then there exists $\varepsilon > 0$ such that for any $x \in B_m(\varepsilon)$, **the geodesic connecting m and x inside $B_m(\varepsilon)$ is unique.**

Step 2: The corresponding subset of M can be identified with a preimage of the ball under the projection, that is, **obtained as an intersection of M and a cylinder $B^n \times \mathbb{R}$**, with the axis $\mathbb{R}$ transversal to M.

Step 3: Any two closed cylinders R^{n+1} intersect either in an open set or a line or a point; in our case this line has to be transversal to M. Since the intersection of two metric spheres in a geodesic space cannot be open, this implies that the intersection of two closed balls $\overline{B}_x(r)$ and $\overline{B}_y(d(x,y) - r)$ is a single point. The same argument as used to prove Hopf-Rinow and Cohn-Vossen theorem implies **the geodesic is unique in a subset which projects to $T_m M$ bijectively.** ■

Geodesics in space forms

COROLLARY: Let $M \subset \mathbb{R}^{n+1}$ be a space form realized as above, and $x, y \in M$ distinct points. **Then the geodesic connecting x to y is a subset of the 1-dimensional manifold $M \cap \langle x, y \rangle$ obtained by intersecting M and a 2-dimensional subspace generated by x, y.**

Proof: The group $\operatorname{St}(x, y) := SO(n-1)$ acts on the set of geodesics connecting x to y. **An orbit of any vector which does not belong to $\langle x, y \rangle$ is not a point**, and the geodesic is unique, hence it belongs to $M \cap \langle x, y \rangle$. ■

Isometry group of a space form acting (almost) bi-transitively

COROLLARY: Let $M = G/H$ be a simply connected space form, and $x_i, y_i \in M$, $i = 1, 2$ two pairs of points which satisfy $d(x_1, y_1) = d(x_2, y_2)$. **Then there exists an isometry $g \in G$ mapping (x_1, y_1) to (x_2, y_2).**

Proof: For $\mathbb{R}^n$, this statement is clear, hence we can assume $M = S^n$ or $M = \mathbb{H}^n$. As above, we realize M as a sphere or a hyperboloid

$$\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \ \mid \ x_0^2 = 1 + \sum x_i^2\}$$

in $\mathbb{R}^{n+1}$. Then the metric sphere in M is given $\{x \in M \ \mid \ g(x, m) = \delta\}$. Therefore, the equality $d(x_1, y_1) = d(x_2, y_2)$ is equivalent to $g(x_1, y_1) = g(x_2, y_2)$. However, $SO(n+1)$ acts on pairs of unit vectors in $\mathbb{R}^{n+1}$ which satisfy the equality $g(x_1, y_1) = g(x_2, y_2)$ transitively. Similarly, $SO(1, n)$ acts on pairs of unit vectors satisfy the equality $g(x_1, y_1) = g(x_2, y_2)$ transitively **(both assertions are proven by complementing these pairs of vectors by $n - 2$ vectors which are orthonormal).** ■

Tance

REMARK: The term "tance" is due to Alexandre Anan'in.

DEFINITION: We realize the hyperbolic space $\mathbb{H}$ as a hyperboloid $\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \ \ | \ \ x_0^2 = 1 + \sum x_i^2\}$ in R^{n+1}. Define **tance** between two points $x, y \in M$ as $\operatorname{ta}(x,y) = g(x,y)^2$, where g is the pairing on $\mathbb{R}^{n+1}$.

REMARK:

$$g(x,y) = \sqrt{1 + \sum_{i=1}^n x_i^2}\sqrt{1 + \sum_{i=1}^n y_i^2} - \sum_{i=1}^n x_i y_i.$$

By Cauchy inequality, $\sum_{i=1}^n x_i y_i \leqslant \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}$. Then $\operatorname{ta}(x,y) \geqslant 1$**, with equality realized if and only if** $x = y$**.**

PROPOSITION: The geodesic distance $d(x,y)$ is a monotonous function of $\operatorname{ta}(x,y)$.

Proof: A pair of points is uniquely up to $\operatorname{Iso}(\mathbb{H}^n)$-action determined by $\operatorname{ta}(x,y)$ and by $d(x,y)$ as we have already shown. A continuos bijection from $\mathbb{R}^{\geqslant 0}$ to $\mathbb{R}^{\geqslant 1}$ is always monotonous. ■

REMARK: Integrating the distance function, it is possible to show that $\cosh(d(x,y))^2 = \operatorname{ta}(x,y)$. **We won't use this result.**

Some low-dimensional Lie group isomorphisms

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in GL(Lie(G)) with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the center Z of G (prove it).

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional real space equipped with a scalar product of signature (1,2), $SO^+(1,2)$ a connected component of unity, and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$, $SO^+(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later today. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, with $\kappa(a,b) := \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence the map $PSL(2,\mathbb{R}) \longrightarrow SO^+(1,2)$ is surjective. It is injective, because its kernel has to act trivially on $\mathfrak{sl}(2,\mathbb{R})$, but the center of this group is trivial. ■

Möbius transforms

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Since the transition function between the charts $1 : z$ and $z : 1$ on $\mathbb{C}P^1$ is $x \longrightarrow x^{-1}$, we have

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z(1 + \sum_{i \geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ■

Hermitian and pseudo-Hermitian forms

DEFINITION: Let (V,I) be a (real) vector space equipped with a complex structure, and h a bilinear symmetric form. It is called **pseudo-Hermitian** if $h(x,y) = h(Ix,Iy)$.

REMARK: The corresponding quadratic form $x \mapsto h(x,x)$ is sometimes writen as $h(x)$. **One can recover $h(x,y)$ from $h(x)$ as usual: $2h(x,y) = h(x+y) - h(x) - h(y)$.**

REMARK: Often one considers a complex-valued form $h(x,y)+\sqrt{-1}h(x,Iy)$. It is **sesquilinear** as a form on the complex space: $h(\lambda x,y) = \lambda(x,y)$, $h(x,\lambda y) = \overline{\lambda}(x,y)$, for any $\lambda \in \mathbb{C}$, and the imaginary part $\sqrt{-1}\, h(x,Iy)$ is anti-symmetric.

CLAIM: Let (V,I,h) be a pseudo-Hermitian vector space. Consider V as a complex vector space, $\dim_{\mathbb{C}} V = n$. Then there exists a basis $z_1, ..., z_n$ in V such that $h(z_i,z_j) = 0$ for $i \neq j$ (such a basis is called **orthogonal**). Moreover, this basis can be chosen in such a way that $h(z_i,z_i)$ is ± 1 or 0 (such a basis is called **orthonormal**).

Orthonormal basis for a pseudo-Hermitian form

CLAIM: For any pseudo-Hermitian form h on (V, I), **there exists orthonormal basis $z_1, ..., z_n$.**

Proof: Use induction on $\dim V$. If $h = 0$, this claim is clear. Assume that $h \neq 0$. For any $A \subset V$, denote by $A^\perp$ the space $\{x \in V \ \mid\ h(x, a) = 0 \forall a \in A\}$.

Choose any $z_1 \in V$ such that $h(z_1, z_1) \neq 0$, and let $z_1^{\perp, \mathbb{C}} := \langle z_1, I(z_1) \rangle^\perp = z_1^\perp \cap I(z_1)^\perp$. This is a complex vector space which is orthogonal to z_1. It can also be obtained as an orthogonal complement with respect to the sesquilinear form $h(x, y) + \sqrt{-1}\, h(x, Iy)$.

By induction assumption, the space $z_1^{\perp, \mathbb{C}}$ has an orthonormal basis $z_2, ..., z_n$. **Then $z_1, ..., z_n$ is an orthogonal basis in V.** Replacing z_1 by $h(z_1, z_1)^{1/2} z_1$, we obtain an orthonormal basis $z_1, ..., z_n$. ∎

Signature of a Hermitian form

REMARK: By Sylvester's law of inertia, the number of z_i such that $h(z_i, z_i) = 1$, $h(z_i, z_i) = -1$ and $h(z_i, z_i) = 0$ **is independent form the choice of an orthonormal basis.**

DEFINITION: Let (V, I, h) be a vector space with non-degenerate Hermitian form, and $z_1, ..., z_n$ an orthonormal basis, $h(z_i, z_i) = 1$ for $i = 1, ...p$ and $h(z_i, z_i) = 1$ for $i = p + 1, ..., n$, with $q = n - p$. Then h is called **Hermitian form of signature (p, q)**. The group of complex linear automorphisms preserving h **is denoted $U(p, q)$.**

Normal form for a pair of Hermitian forms

Theorem 1: Let $V = \mathbb{R}^n$, and $h, h' \in \mathsf{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

Theorem 1': Let $V = \mathbb{C}^n$, and $h, h' \in \mathsf{Sym}^2 V^*$ be two (pseudo-)Hermitian forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

REMARK: In this basis, h' is written as diagonal matrix, with eigenvalues $\alpha_1, ..., \alpha_n$ independent from the choice of the basis. Indeed, consider h, h' as maps from V to V^*, $h(v) = h(v, \cdot)$. Then $h_1 h^{-1}$ is an endomorphism with eigenvalues $\alpha_1, ..., \alpha_n$. **This implies that Theorem 1 gives a normal form of the pair h, h'.**

Circles on a sphere

DEFINITION: A circle in S^2 is an orbit of rotation subgroup, that is, a subgroup $U \subset SO(3) = PU(2) \subset PGL(2, \mathbb{C})$ isomorphic to S^1 and acting on $S^2 = \mathbb{C}P^1$ by isometries.

REMARK: Let U be a rotation group rotating S^2 around an axis passing through x and $y \in S^2$. **Any orbit C of U satisfies $d(x, v) = const$ for all $v \in C$.**

LEMMA: Let z_1, z_2 be a basis in $V = \mathbb{C}^2$, and $h(az_1 + bz_2) = \alpha|a|^2 - \beta|b|^2$ a pseudo-Hermitian form, with $\alpha, \beta \geqslant 0$. **Then the set $Z_h = \mathbb{P}\{x \in V \ | \ h(x) = 0\}$ is a circle in $\mathbb{C}P^1$, and all circles can be obtained this way.**

Proof: In homogeneous coordinates, Z_h is the set of all $x : y$ such that $\alpha|x|^2 = \beta|y|^2$, and rotation acts as $x : y \longrightarrow x : e^{\sqrt{-1}\,\theta}y$. Clearly, the orbits of rotation are precisely the sets Z_h for different α, β. ■

Orbits of compact one-parametric subgroups in $PGL(2,\mathbb{C})$

LEMMA: Let $G \cong S^1$ be a compact 1-dimensional subgroup in $PGL(2,\mathbb{C})$. **Then any G-orbit in $\mathbb{C}P^1$ is a circle.**

Proof: Let $V = \mathbb{C}^2$, and consider the natural projection map

$$\pi : SL(V) \longrightarrow PGL(2,\mathbb{C}) = SL(V)/\pm 1.$$

Then $\tilde{G} = \pi^{-1}(G)$ is compact. Chose a $\tilde{G}$-invariant Hermitian metric h on V by averaging a given metric with $\tilde{G}$-action. By definition, circles on $\mathbb{C}P^1$ are orbits of rotation subgroups in $U(V,h)$. **Since $u(\tilde{G})$ is a 1-dimensional compact subgroup in $U(V,h)$, its orbit is a circle.** ■

COROLLARY: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.
■

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

Proof: By Cauchy formula, $f(0) = \frac{1}{2\pi} \int_{\partial\Delta} f(z) \frac{dz}{-\sqrt{-1}\, z}$, where Δ is a disk in $\mathbb{C}$. An elementary calculation gives $\frac{dz}{-\sqrt{-1}\, z}|_{\partial\Delta} = \operatorname{Vol}(\partial\Delta)$ – the volume form on $\partial\Delta$. Therefore, $f(0)$ is the average of $f(z)$ on the circle, and it is the average of $f(z)$ on the disk Δ. Now, absolute value of the average $|\operatorname{Av}_{x\in S} \mu(x)|$ of a complex-valued function μ on a set S is equal to $\max_{x\in S} |\mu(s)|$ only if $\mu = const$ almost everywhere on S **(check this)**. ■

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of $PGL(2, \mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi: G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\operatorname{St}_x(G_1) = \operatorname{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker\Psi$ belongs to $\operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk

REMARK: The group $PU(1,1)\subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l\in \mathbb{C}P^1 \ \mid\ h(l,l)>0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta\subset\mathbb{C}$ be the unit disk, Aut(Δ) the group of its conformal automorphisms, and $\Psi:\ PU(1,1)\longrightarrow$ Aut(Δ) the map constructed above. **Then Ψ is an isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\mathrm{St}_x(PU(1,1))=S^1$ and $\mathrm{St}_x(\mathrm{Aut}(\Delta))=S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma **(prove it!)**. ■

COROLLARY: Let h be a homogeneous metric on $\Delta=PU(1,1)/S^1$. **Then (Δ,h) is conformally equivalent to $(\Delta,$ flat metric$)$.**

Proof: The group Aut(Δ) $=PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique up to a contant multiplier **(prove it)**. ■

Upper half-plane

REMARK: The map $z \longrightarrow -(z-\sqrt{-1}\,)^{-1} - \frac{\sqrt{-1}}{2}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}^2$ **(prove it)**.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ **acts on the upper half-plane** $\mathbb{H}^2$ **as** $z \xrightarrow{A} \frac{az+b}{cz+d}$**, where** $a, b, c, d \in \mathbb{R}$**, and** $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$**.**

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}^2$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

COROLLARY: The group of conformal automorphisms of $\mathbb{H}^2$ acts on $\mathbb{H}^2$ preserving a unique, up to a constant, Riemannian metric. **The Riemannian manifold $PSL(2,\mathbb{R})/S^1$ obtained this way is isometric to a hyperbolic space.**

Upper half-plane as a Riemannian manifold

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}^2$. **Then the Riemannian structure s on $\mathbb{H}^2$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}^2$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2 + dy^2)$, where $\mu \in C^\infty(\mathbb{H}^2)$. **It remains to find μ, using the fact that s is $PSL(2,\mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $z \longrightarrow z + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the homothety $H_\lambda(z) = \lambda z$ also fixes s; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2(dx^2 + dy^2)$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ■

Geodesics in Poincaré half-plane

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2, \mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}^2$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and l the vertical line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}^2$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}^2$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}^2) = PSL(2, \mathbb{R})$ ■

Geodesics in Poincaré half-plane (2)

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■