Metric spaces

lecture 13: Hyperbolic space is CAT(0)

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Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/O(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms (reminder)

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: **From now on, all space forms are assumed to be homogeneous Riemannian manifolds**.

COROLLARY: Let $M \subset \mathbb{R}^{n+1}$ be a space form realized as above, and $x, y \in M$ distinct points. **Then the geodesic connecting x to y is a subset of the 1-dimensional manifold $M \cap \langle x, y\rangle$ obtained by intersecting M and a 2-dimensional subspace generated by x, y.**

COROLLARY: Let $M = G/H$ be a simply connected space form, and $x_i, y_i \in M$, $i = 1, 2$ two pairs of points which satisfy $d(x_1, y_1) = d(x_2, y_2)$. **Then there exists an isometry $g \in G$ mapping (x_1, y_1) to (x_2, y_2).**

Tance (reminder)

REMARK: The term "tance" is due to Alexandre Anan'in.

DEFINITION: We realize the hyperbolic space $\mathbb{H}$ as a hyperboloid $\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \ \mid \ x_0^2 = 1 + \sum x_i^2\}$ in R^{n+1}. Define **tance** between two points $x, y \in M$ as $\operatorname{ta}(x,y) = g(x,y)^2$, where g is the pairing on $\mathbb{R}^{n+1}$.

REMARK:

$$g(x,y) = \sqrt{1 + \sum_{i=1}^n x_i^2}\sqrt{1 + \sum_{i=1}^n y_i^2} - \sum_{i=1}^n x_i y_i.$$

By Cauchy inequality, $\sum_{i=1}^n x_i y_i \leqslant \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}$. Then $\operatorname{ta}(x,y) \geqslant 1$**, with equality realized if and only if** $x = y$**.**

PROPOSITION: The geodesic distance $d(x,y)$ is a monotonous function of $\operatorname{ta}(x,y)$.

Proof: A pair of points is uniquely up to $\operatorname{Iso}(\mathbb{H}^n)$-action determined by $\operatorname{ta}(x,y)$ and by $d(x,y)$ as we have already shown. A continuos bijection from $\mathbb{R}^{\geqslant 0}$ to $\mathbb{R}^{\geqslant 1}$ is always monotonous. ■

REMARK: Integrating the distance function, it is possible to show that $\cosh(d(x,y))^2 = \operatorname{ta}(x,y)$. **We won't use this result.**

Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points on a geodesic metric space (M, d), and $r = d(a, b)$, and $\gamma : [0, r] \longrightarrow M$ a minimizing geodesic connecting a to b. Consider the function $d_c : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ taking t to $d(c, \gamma(t))$. Let $\triangle(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be the comparison triangle, and $d_{\overline{c}} : [0, r] \longrightarrow \mathbb{R}^{\geqslant 0}$ the function taking t to $d(\overline{c}, \overline{\gamma}(t))$, where $\overline{\gamma} : [0, r] \longrightarrow \mathbb{R}^2$ is the side of the comparison triangle with the normal parametrization. The function $d_{\overline{c}}$ is called **the comparison function.**

DEFINITION: An intrinsic metric space (M, d) has **non-negative curvature** if any point has a neighbourhood V with intrinsic metric such that for any geodesic triangle in V, one has $d_c \geqslant d_{\overline{c}}$, and has **non-positive curvature** if $d_c \leqslant d_{\overline{c}}$. An geodesic metric space M is called **CAT(0)-space**, after Elie Cartan, A. D. Alexandrov and V. A. Toponogov, if the inequality $d_c \leqslant d_{\overline{c}}$ holds for all geodesic triangles.

REMARK: Intrinsic metric spaces with these curvature restrictions are called **Alexandrov spaces.**

PROPOSITION: Let $z \in M$, where M is a CAT(0)-space. **Then the function $d_z(x) := d(z, x)$ is convex.**

Angles in Alexandrov spaces (reminder)

DEFINITION: Let a, b, c be points in a metric space (M, d). **A comparison triangle** $\triangle(\overline{a}, \overline{b}, \overline{c})$ is a triangle in $\mathbb{R}^2$, with vertices $\overline{a}, \overline{b}, \overline{c}$, and side lengths $|\overline{a}, \overline{b}| = d(a, b)$, $|\overline{a}, \overline{c}| = d(a, c)$, $|\overline{b}, \overline{c}| = d(b, c)$. **This triangle exists, and is uniquely determined, up to an isometry,** (the existence follows from the triangle inequality). The angle $\measuredangle(\overline{a}, \overline{b}, \overline{c}) \in [0, \pi]$ in the triangle $\overline{a}, \overline{b}, \overline{c}$ is denoted $\theta(a, b, c)$; it is called **the comparizon angle**.

DEFINITION: Let a, b, c be three points on a metric space, and $\triangle(\overline{a}, \overline{b}, \overline{c})$ the comparison triangle. Consider the minimizing geodesics γ_1, γ_2, connecting a to b and a to c. **The angle comparison condition for non-positive curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \leqslant \measuredangle(\overline{c}\overline{a}\overline{b})$. **The angle comparison condition for non-negative curvature** is inequality $\measuredangle(\gamma_1, a, \gamma_2) \geqslant \measuredangle(\overline{c}\overline{a}\overline{b})$ together with the equality $\measuredangle(\gamma_+, p, \mu) + \measuredangle(\gamma_-, p, \mu) = \pi$ for any two adjacent angles $\measuredangle(\gamma_+, p, \mu)$, $\measuredangle(\gamma_-, p, \mu)$.

THEOREM: The angle comparison condition is equivalent to the Alexandrov condition $d_{\overline{c}} \leqslant d_c$ (or $d_{\overline{c}} \geqslant d_c$) for the same sign of the curvature.

Sum of the angles

REMARK: From the angle comparizon it follows immediately that **the sum of the angles of any geodesic triangle in CAT(0)-space is $\leqslant \pi$.** Indeed, from CAT(0) it follows that the angles in $\triangle(abc)$ are all $\leqslant$ than the angles in $\triangle(\overline{a}\overline{b}\overline{c})$, and the sum of those is π because $\triangle(\overline{a}\overline{b}\overline{c}) \subset \mathbb{R}^2$.

Lagrange has shown that the axioms of Euclide (without the parallel postulate) **imply that the sum of the angles of any geodesic triangle on a plane is $\leqslant \pi$.**

We will prove that **the hyperbolic space satisfies CAT(0)-conditions.**

The positive cone

Let g denote the standard signature $(1,n)$ scalar product on $V = \mathbb{R}^{n+1} = \mathbb{R}^{1,n}$, $g(v,v) = v_0^2 = \sum_{i=1}^n v_i^2$. A vector $v \in V$ is called **positive** if $g(v,v) > 0$ and the 0-th coordinate of v is positive. Clearly, $(x_0, ..., x_n)$ is positive if and only if $x_0^2 \geqslant \sum_{i=1}^n x_i^2 > 0$ and $x_0 > 0$. By Cauchy inequality,

$$\sum_{i=1}^n (x_i + y_i)^2 = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 + 2\sum_{i=1}^n x_i y_i \leqslant$$
$$\leqslant \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 + 2\sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2} \leqslant \left(\sqrt{\sum_{i=1}^n x_i^2} + \sqrt{\sum_{i=1}^n y_i^2}\right)^2,$$

hence for any positive vectors x, y, their sum satisfies

$$(x_0 + y_0)^2 \geqslant \left(\sqrt{\sum_{i=1}^n x_i^2} + \sqrt{\sum_{i=1}^n y_i^2}\right)^2 \geqslant \sum_{i=1}^n (x_i + y_i)^2.$$

DEFINITION: We have proven that **the set of positive vectors in V is a convex cone,** called **positive cone**. **For any positive x, p, the vector $x + p$ is positive, hence $x - p$ which is orthogonal to $x + p$ has negative square.** Indeed, g is negative definite on an orthogonal complement to a positive vector, because its signature is $(1,n)$.

Reflections in a positive cone

The reflection in a hyperplane.
Consider a vector $u \in V$ which satisfies $g(v,v) = -2$, and let $r_v : \ u \mapsto u + g(u,v)v$. Then $g(r_v(u), r_v(u)) = g(u,u) + g(v,v)g(u,v)^2 + 2g(u,v)^2 = g(u,u)$, hence it is an isometry. It acts as identity on $v^\perp$ and takes v to $-v$, hence it is a reflection. To find v such that $r_v(p) = x$, we write $x - p = g(p,v)v$, which gives $v = 2\sqrt{-g(x-p, x-p)}(x-p)$. The space $v^\perp$ intersects with the plane $\langle p, x\rangle$ in the line $\mathbb{R}\cdot(x+p)$, and the corresponding point of M is $\sqrt{g(x+p, x+p)^{-1}}(x+p)$. **This is the midpoint between x and p.**

Central symmetry.
Let $v \in V$ be a vector which satisfies $g(v,v) = 2$, and $c_v(u) := (u,v)v - u$. This map is an isometry, because $g(c_v(u), c_v(u)) = g(u,u) + g(v,v)g(u,v)^2 - 2g(u,v)^2 = g(u,u)$; it fixes v and acts on $T_vM = V^\perp$ as $-\operatorname{Id}$, hence $u \mapsto c_v(u)$ **is a central symmetry with center in $2^{-1/2}v \in M$.**

COROLLARY: The central symmetry c_v **reflects any geodesic passing through v to itself.**

Proof: Indeed, geodesics correspond to 2-dimensional planes $L \subset V$, and those are preserved by reflections with center in L. ■

Hyperbolic spaces are CAT(0)

THEOREM: The CAT(0) angle comparison condition holds in the hyperbolic space $\mathbb{H}^n$.

Proof: Let $\langle p, x\rangle \cap M$ and $\langle p, y\rangle \cap M$ be geodesics. We find the midpoints x' between p and x and y' between p and y. Then we take the next midpoint x'' between x' and p and y'' between y' and p, and so on. Clearly, $\measuredangle(xpy) = \lim_n \theta(x^{(n)}py^{(n)})$, where $x^{(n)}$ is the n-th midpoint. Therefore, the angle comparison property $\measuredangle(xpy) \leqslant \theta(xpy)$ would follow from $\theta(x'py') \leqslant \theta(xpy)$.

Consider the comparison triangles $\triangle(\overline{xpy})$ and $\triangle(\overline{x}'\overline{py}')$; by definition, $\theta(x'py')$ is the angle at p in $\triangle(\overline{x}'\overline{py}')$, and $\theta(xpy)$ is the angle at p in $\triangle(\overline{xpy})$. Since $|\overline{xp}| = 2|\overline{x}'\overline{p}|$ and $|\overline{yp}| = 2|\overline{y}'\overline{p}|$, the inequality $\theta(x'py') \leqslant \theta(xpy)$ is equivalent to $|\overline{x}', \overline{y}'| \leqslant 2|\overline{x}, \overline{y}|$, **which would imply the angle comparison and hence the CAT(0)-property.**

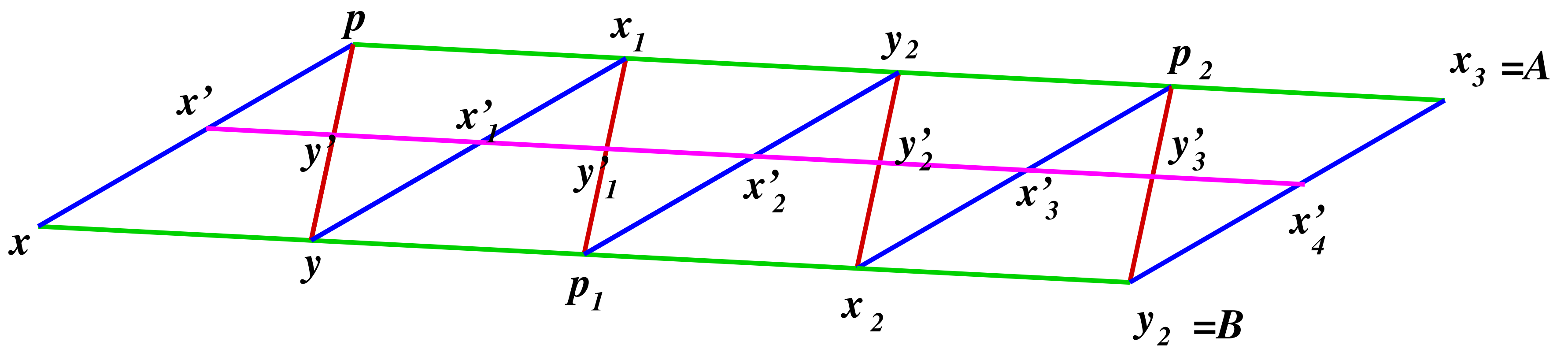
We reduced the CAT(0) property to the following lemma of hyperbolic geometry. Note that **any geodesic triangle in $\mathbb{H}^n$ belongs to a hyperbolic plane $H^2 \subset H^n$,** which is easily seen because points are vectors in V and geodesics are subspaces connecting these points.

LEMMA: Let $\triangle(xpy)$ be a triangle in $\mathbb{H}^2$, and x', y' the midpoints between p and x and between p and y. **Then $d(x', y') \leqslant \frac{1}{2}d(x, y)$.**

The midpoints in a hyperbolic triangle

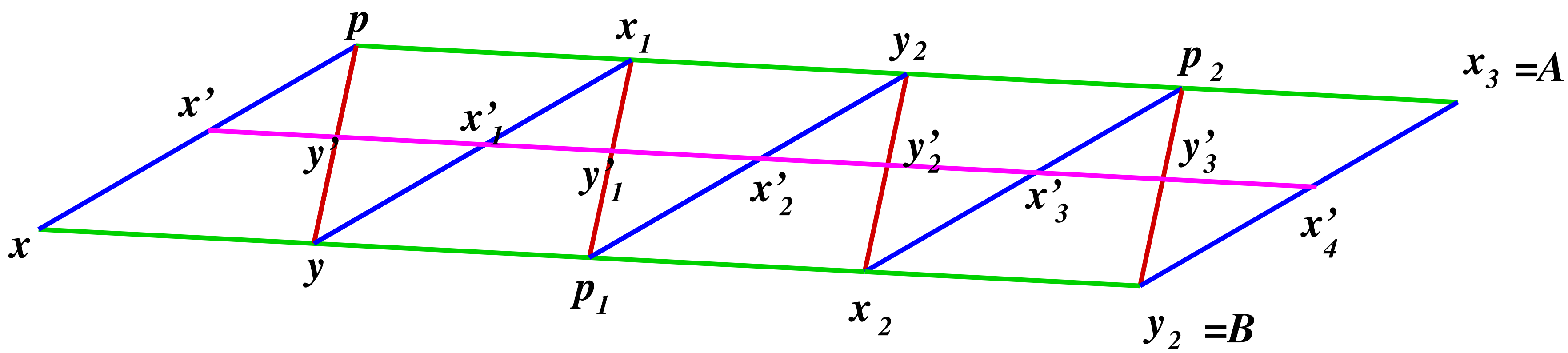
LEMMA: Let $\triangle(xpy)$ be a triangle in $\mathbb{H}^2$, and x', y' the midpoints between p and x and between p and y. **Then $d(x',y') \leqslant \frac{1}{2}d(x,y)$.**

Proof. Step 1: Apply the central symmetry centered in y' to the triangle $\triangle(xpy)$, then the central symmetry in the image of x' and so on, as shown on the picture. This gives a sequence of congruent triangles $\triangle(xpy)$, $\triangle(pyx_1)$, $\triangle(yx_1p_1)$, with the midlines $[x'y']$, $[y'x'_1]$, and so on.



Since the central symmetry preserves a geodesic passing through its center, the midpoints $x', y', x'_1, y'_1, ...$ sit on the same line.

The midpoints in a hyperbolic triangle (2)



Suppose we have drawn $2n$ triangles; the last midpoint is x'_n. Let A be the point in the top right corner. **The length of $[x', x'_n]$ is $2nd(x', y')$, because all points x'_i and y'_i belong to the same geodesic, and there are $2n$ equal segments of length $d(x', y')$.**

The top green line connecting p and A has n segments of length $d(x, y)$, which gives $d(p, A) \leqslant nd(x, y)$. Clearly, $d(x', p) = \frac{1}{2}d(x, p)$ and $d(A, x'_n) = \frac{1}{2}d(x, p)$

The triangle inequality associated with the polygonal chain x', p, A, x'_n gives

$$2nd(x', y') = d(x', x'_n) \leqslant \frac{1}{2}d(x, p) + nd(x, y) + \frac{1}{2}d(x, p) = nd(x, y) + d(x, p).$$

Dividing by n and passing to the limit $n \to \infty$, we obtain

$$2d(x', y') \leqslant \lim_{n \to \infty} d(x, y) + \frac{1}{n}d(x, p) = d(x, y).$$

■

Area in a hyperbolic plane

COROLLARY: An area of an n-sided polygon in a hyperbolic plane is equal to $\pi(n-2) - \sum \alpha_i$, where α_i are its plane angles.

Proof: This formula defies an additive, isometry invariant function $V: \mathfrak{B} \longrightarrow \mathbb{R}^{\geqslant 0}$ on the Boolean algebra $\mathfrak{B}$ of all polygons in $\mathbb{H}^2$. Its additivity follows from an inductive argument where we cut a polygon A onto two pieces A_1 and A_2 by a line and show that $V(A) = V(A_1) + V(A_n)$; induction by a number of lines implies additivity for any cut.

It takes non-negative values on any triangle (and, by additivity, on any polygon) because a sum of the angles of a triangle is $\leqslant \pi$, which follows from CAT(0)-property.

Now, from the first lectures on measure theory it follows that any additive, isometry-invariant function $V: \mathfrak{B} \longrightarrow \mathbb{R}^{\geqslant 0}$ is proportional to the volume (the standard argument which proves this assertion is valid not only for the Euclidean plane, but for any homogeneous Riemannian manifold). ■