Metric spaces

lecture 14: Gromov hyperbolic spaces

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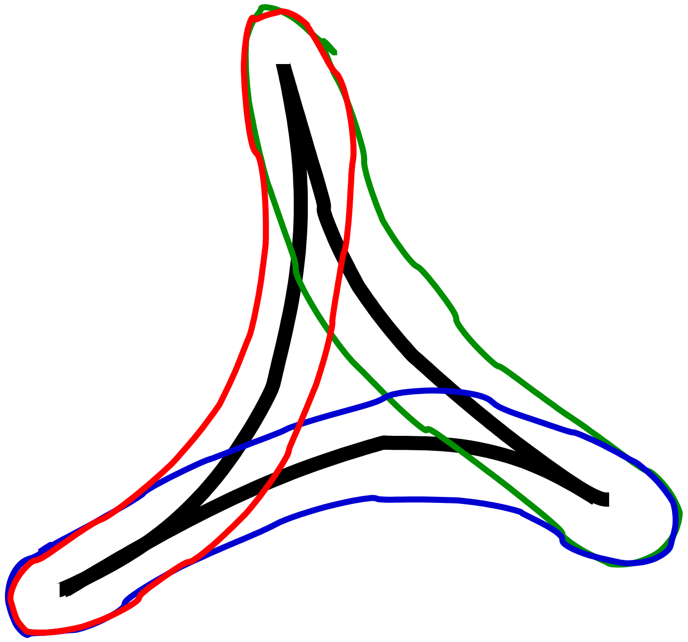
Thin triangles

DEFINITION: Let $\triangle(abc)$ be a geodesic triangle in a metric pace, with the sides denoted as $[a,b],[b,c]$ and $[c,a]$. **The minsize** of of a triangle is the infimum of all δ such that each side belongs to the δ-neighbourhood of the other two:

$$[a,b] \subset [b,c](\delta) \cup [c,a](\delta), \quad [a,c] \subset [b,c](\delta) \cup [a,b](\delta), \quad [b,c] \subset [b,c](\delta) \cup [a,b](\delta),$$

where $X(\delta)$ denotes the δ-neighbourhood. A triangle is called **δ-slim** (in the sense of Rips), if its minsize is $\leqslant \delta$.



Hyperbolic spaces

DEFINITION: A geodesic metric space is called **δ-hyperbolic** in the sense of Rips if all its geodesic triangles are δ-hyperbolic. We say that X is **Gromov hyperbolic**, if it is δ-hyperbolic for some constant δ.

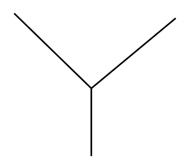
REMARK: There are many definitions of hyperbolicity which differ by the constant. The constant δ itself does not matter, when someone says "the definition A of hyperbolicity is equivalent to B", this just means that **for some number $C > 0$, δ-hyperbolicity in the sense of A implies $C\delta$-hyperbolicity in the sense of B, and δ-hyperbolicity in the sense of B implies $C\delta$-hyperbolicity in the sense of A.**

The model tripod

DEFINITION: A metric tree is the metric space of a graph which is simply connected and connected.

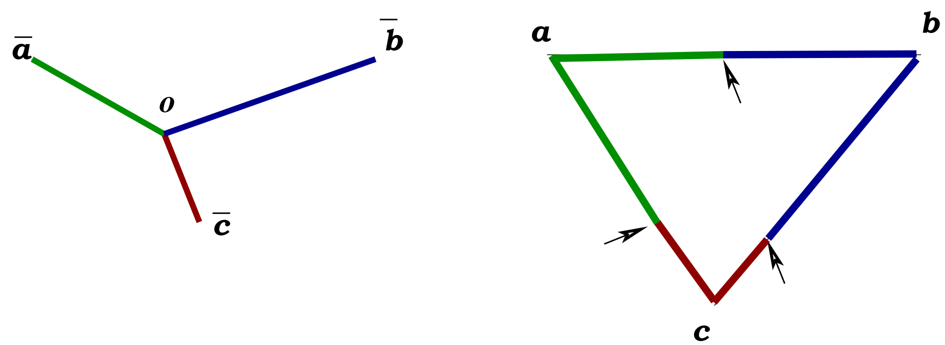
CLAIM: A tree is Rips 0-hyperbolic.

Proof: Let $\triangle(abc)$ be a geodesic triangle. Since the minimizers are unions of segments of graph components, $\triangle(abc)$ **is a connected subgraph, that is, a tree:**



It is Rips 0-hyperbolic, which is seen from the picture. ■

DEFINITION: Let $\triangle(abc)$ be a geodesic triangle. Define **a model 0-hyperbolic triangle**, or **a model tripod** as a tree $\triangle(\bar{a}\bar{b}\bar{c})$ with three free ends



and three edges, connected in a fourth vertex, such that the corresponding distances are equal: $|ab| = |\overline{a}\overline{b}|$, $|ac| = |\overline{ac}|$, $|bc| = |\overline{b}\overline{c}|$.

The comparison map

CLAIM: Let $\triangle(abc)$ be a geodesic triangle in a metric space, and $\triangle(\overline{a}\overline{b}\overline{c})$ the model tripod. **Then there exists a unique map $\Psi: \triangle(abc) \longrightarrow \triangle(\overline{a}\overline{b}\overline{c})$ which defines an isometry on each side, and takes the vertices of $\triangle(\overline{a}\overline{b}\overline{c})$ to free vertices.**

Proof: The model tripod is made of three intervals of length $|\overline{a}o| = (b,c)_a =$, $|\overline{c}o| = (a,b)_c$ and $|\overline{b}o| = (a,c)_b$ has sides which are pairwise sums of Gromov products, such as

$$(b,c)_a + (a,b)_c = \frac{1}{2}(|ab| + |ac| - |bc| + |ac| + |bc| - |ab|) = |ac|. \quad \blacksquare$$

DEFINITION: This map is called **The comparison map.**

DEFINITION: Let $\varphi: X \longrightarrow Y$ be a map of metric spaces (not necessarily continuous). The **codiameter** $\operatorname{codiam}\varphi$ is defined as

$$\operatorname{codiam}(\varphi) := \sup_{a,b\in X} |d(x,y) - d(\varphi(x),\varphi(y))|.$$

It measures how far φ is from an isometry.

PROPOSITION: Let $\Psi : \triangle(abc) \longrightarrow \triangle(\overline{a}\overline{b}\overline{c})$ be the comparison map to the model tripod defined above. Then

(a) **If $\operatorname{codiam}\Psi \leqslant \delta$, the triangle $\triangle(abc)$ is δ-slim.**

(b) **If $\triangle(abc)$ is δ-slim, then $\operatorname{codiam}\Psi \leqslant 2\delta$.**

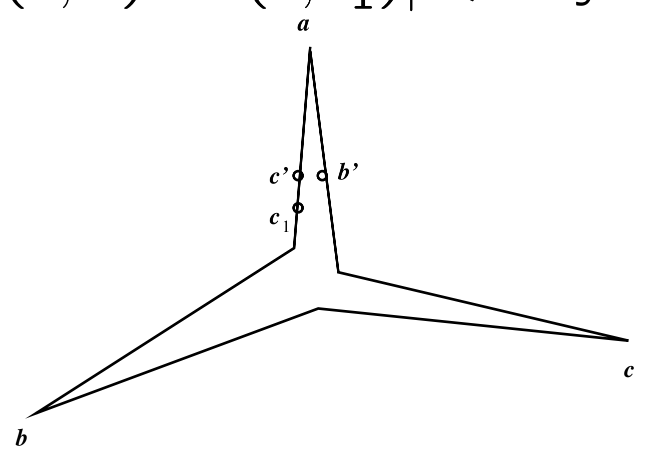
Codiameter of the comparison map

PROPOSITION: Let $\Psi: \triangle(abc) \longrightarrow \triangle(\overline{a}\overline{b}\overline{c})$ be the map to the model tripod. Then

(a) **If $\operatorname{codiam}\Psi \leqslant \delta$, the triangle $\triangle(abc)$ is δ-slim.**

(b) **If $\triangle(abc)$ is δ-slim, then $\operatorname{codiam}\Psi \leqslant 2\delta$.**

Proof. Step 1: (a) is clear. To prove (b), consider $b' \in [ab]$, $c', c_1 \in [ac]$. Clearly, $|b'c_1| < \delta$ implies $|d(a,b') - d(a,c_1)| < \delta$ by the triangle inequality.



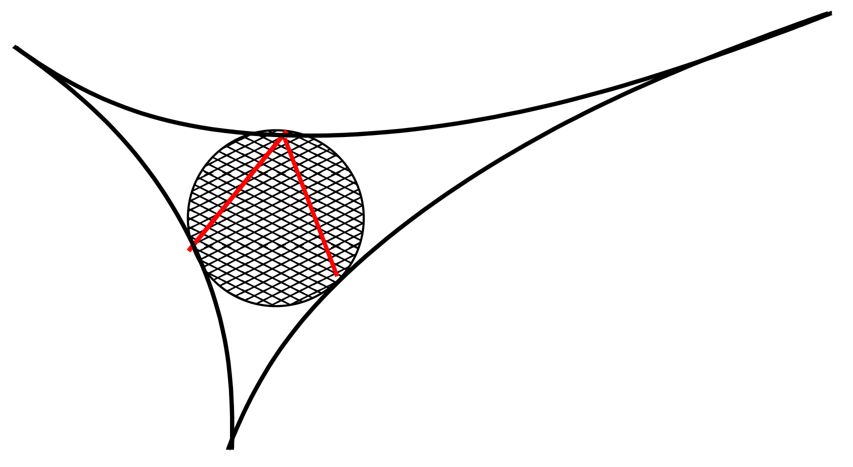
Step 2: Let $d(b', [ac]) < \delta$, $|ac'| = |ab'|$. These are points which are mapped to the same point under the comparison map φ. Let $c_1 \in [ac]$ be a point which satisfies $d(c_1, b') \leqslant \delta$. Step 1 implies $|d(a,b') - d(ac_1)| < \delta$, hence $d(c_1, c') < \delta$, and then $d(c', b') < 2\delta$. This implies that $\operatorname{codiam}\varphi < 2\delta$. ■

Hyperbolicity of a hyperbolic space form

THEOREM: The hyperbolic space form $\mathbb{H}^n$ is Gromov hyperbolic.

Proof. Step 1: Since any geodesic triangle belongs to a hyperbolic plane, **it suffices to prove it for $\mathbb{H}^2$.**

Step 2: Let $\triangle(abc)$ be a geodesic triangle in $\mathbb{H}^2$, and B an inscribed circle. On each side of the triangle, a maximum of the distance to the union of other two sides is bounded by the distance by the points where B is tangent to the sides.



This implies that the minsise of $\triangle(abc)$ satisfies $T(abc) < 2R$, where R is the radius of the inscribed circle.

Hyperbolicity of a hyperbolic space form (2)

Step 2: the minsise of $\triangle(abc)$ satisfies $T(abc) < 2R$, where R is the radius of the inscribed circle.

Step 3: The area of a circle of radius R grows with R indefinitely. Indeed, you can tile a hyperbolic plane by infinitely many conrguent hexagons, hence its area is infinite.

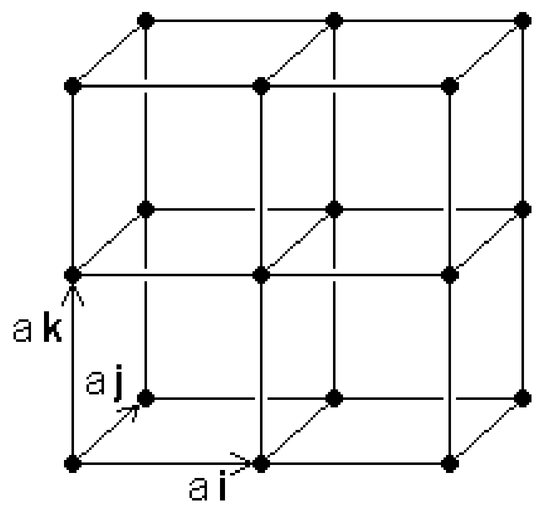
Step 4: An area of an n-sided polygon is equal to $\pi(n-2) - \sum \alpha_i$, where α_i are its plane angles. In particular, the area of a triangle is $\leqslant \pi$. **Therefore a radius of an inscribed circle is bounded.**

Cayley graph

DEFINITION: A set of generators of a group G is a set $S \subset G$ generating G multiplicatively. **We would always assume that $s \in S \Leftrightarrow s^{-1} \in S$.**

DEFINITION: Let G be a group, and $\{s_i\}$ a collection of generators. **The Cayley graph** of the pair $(G, \{s_i\})$ is the metric graph, with the set of vertices identified with G, and edges connecting g and gs_i. The length of all edges the Cayley graph is set to the same number t, usually $t = 1$.

EXAMPLE: The Cayley graph for $\mathbb{Z}^n$ with the standard set of generators is a cubic lattice.



Hyperbolic groups

DEFINITION: Let Γ be a group and S its generator set. We say that Γ is **Gromov hyperbolic** if its Cayley graph is δ-hyperbolic, for some $\delta > 0$.

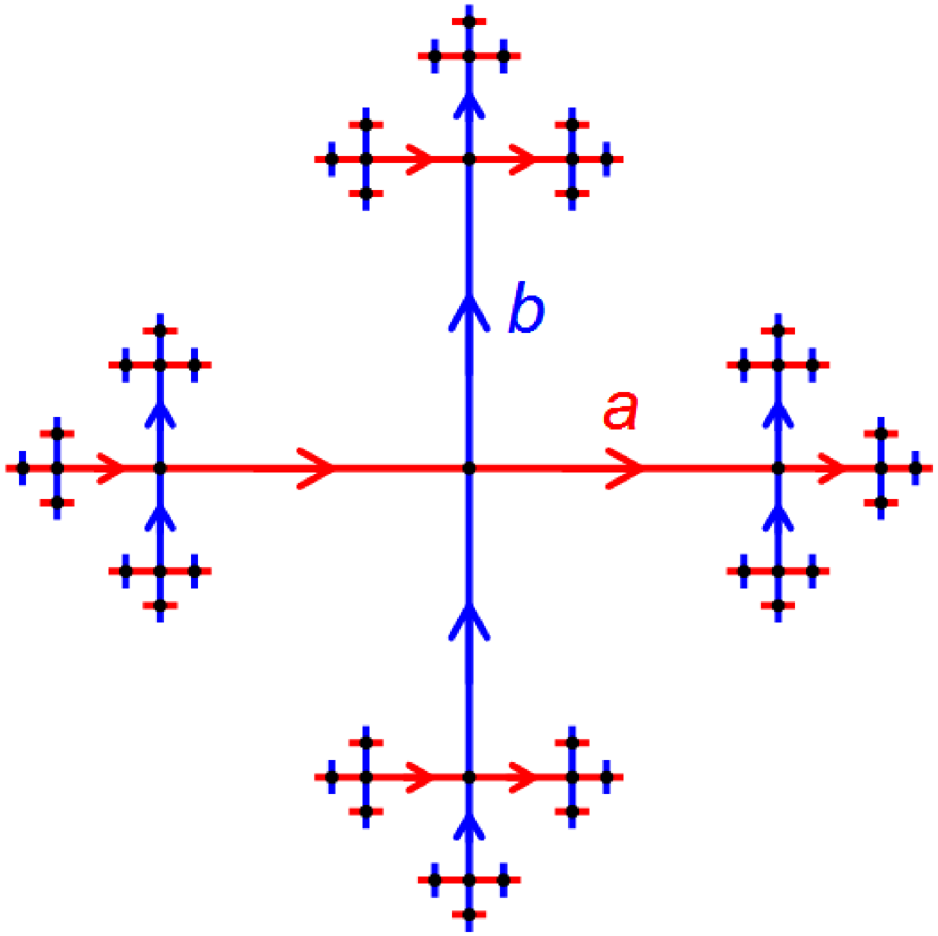
DEFINITION: A group G is **free** if it isomorphic to the fundamental group of a bouquet of circles.

DEFINITION: **The free product** $(\mathbb{Z}/n_1\mathbb{Z})*(\mathbb{Z}/n_2\mathbb{Z})*...*(\mathbb{Z}/n_k\mathbb{Z})$ is the group with generators $x_1, ..., x_k$ and relaions $x_1^{n_1} = 1, x_2^{n_2} = 1, ..., x_k^{n_k} = 1$.

EXERCISE: Prove that **the group $(\mathbb{Z}/n_1\mathbb{Z})*(\mathbb{Z}/n_2\mathbb{Z})*...*(\mathbb{Z}/n_k\mathbb{Z})$ is always hyperbolic.**

Cayley graph for a free group

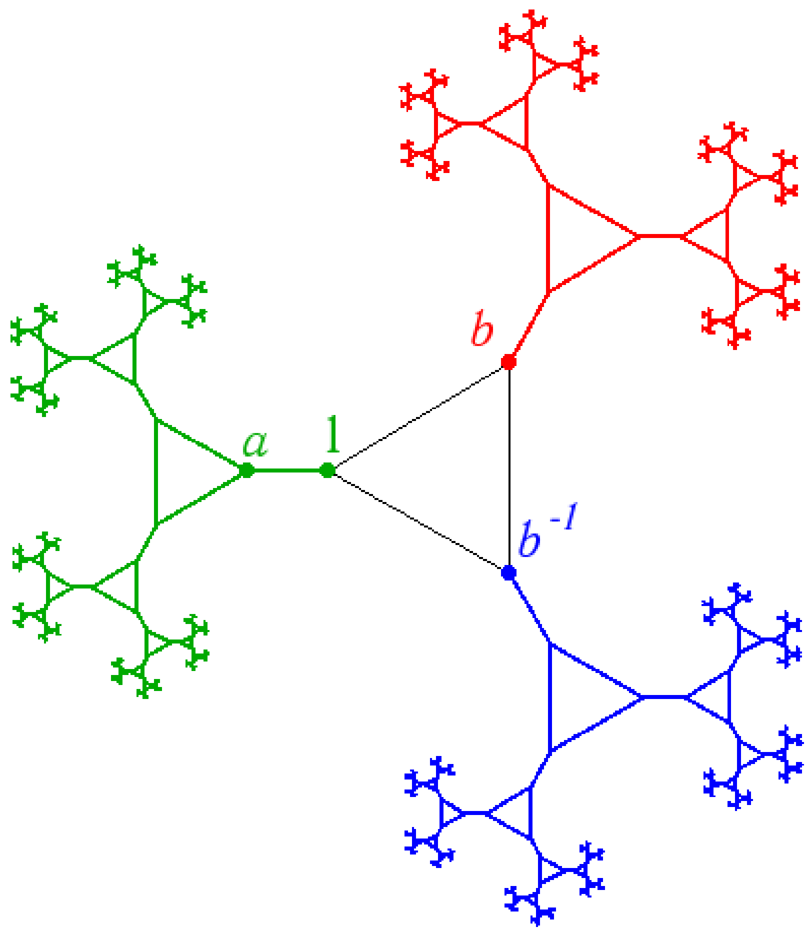
EXAMPLE: Cayley graph for a free group is a regular tree



Cayley graph for a free group $\mathbb{F}_2$ generated by a, b.

CLAIM: This graph **is simply connected, hence it is 0-hyperbolic.**

Cayley graph for $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$



Cayley graph for $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

EXERCISE: Prove that this Cayley graph **is hyperbolic, but not simply connected.**