Metric spaces

lecture 15: Gromov product

Misha Verbitsky

IMPA, sala 236

February 3, 2022, 17:00

The Gromov product

DEFINITION: Let $p \in X$ be a point in a metric space. **The Gromov product** $(a,b)_p$ is defined $(a,b)_p := 1/2(|ap| + |bp| - |ab|)$. **It measures for how long the geodesics from p to a and b stay together.**

REMARK: The distance function can be recovered from $(a,b)_p$. Indeed, $(a,a)_p = |ap|$, hence $|ab| = (a,a)_p + (b,b)_p - 2(a,b)_p$.

It is possible to define the distance in terms of the Gromov product.

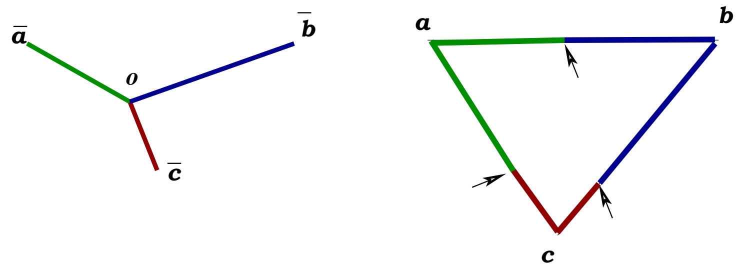
DEFINITION: Let X be a set, and $p \in X$. We say that the function $(\cdot,\cdot)_p : X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ **satisfies the axiom of Gromov product** if the following conditions are satisfied:

[It is symmetric:] $(a,b)_p = (b,a)_p$.
[Non-degenerate:] $(a,a)_p = (a,b)_p = (b,b)_p \Leftrightarrow a = b$.
[Triangle inequality for Gromov product:] $(a,b)_p + (b,c)_p \leqslant (a,c)_p + (b,b)_p$.

CLAIM: Let $(a,b)_p$ is a function $X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ which satisfies the axioms of the Gromov product. **Then $d(a,b) := (a,a)_p + (b,b)_p - 2(a,b)_p$ is a metric on X.** Without the non-degeneracy, this formula defines a pseudometric. ■

The comparison map

DEFINITION: Let $\triangle(abc)$ be a geodesic triangle. Define **a model 0-hyperbolic triangle**, or **a model tripod** as a tree $\triangle(\bar{a}\bar{b}\bar{c})$ with three free ends



and three edges, connected in a fourth vertex, such that the corresponding distances are equal: $|ab| = |\bar{a}\bar{b}|$, $|ac| = |\bar{a}\bar{c}|$, $|bc| = |\bar{b}\bar{c}|$.

CLAIM: Let $\triangle(abc)$ be a geodesic triangle in a metric space, and $\triangle(\bar{a}\bar{b}\bar{c})$ the model tripod. **Then there exists a unique map $\Psi : \triangle(abc) \longrightarrow \triangle(\bar{a}\bar{b}\bar{c})$ which defines an isometry on each side, and takes the vertices of $\triangle(\bar{a}\bar{b}\bar{c})$ to free vertices.**

Proof: The model tripod is made of three intervals of length $|\bar{a}o| = (b,c)_a =$, $|\bar{c}o| = (a,b)_c$ and $|\bar{b}o| = (a,c)_b$ has sides which are pairwise sums of Gromov products, such as

$$(b,c)_a + (a,b)_c = \frac{1}{2}(|ab| + |ac| - |bc| + |ac| + |bc| - |ab|) = |ac|. \quad \blacksquare$$

DEFINITION: This map is called **the comparison map.**

Gromov product and the distance to the geodesic.

PROPOSITION 1: Let $\triangle(abp)$ be a δ-slim triangle. **Then** $d(p,[ab]) \geqslant (a,b)_p \geqslant d(p,[ab]) - 2\delta.$

Proof. Step 1: Let c be the point of $[ab]$ closest to p. The triangle inequality gives $|ap|-|cp|+|bp|-|cp| \leqslant |ac|+|cb| = |ab|$. **This implies** $|ap|+|bp|-|ab| \leqslant 2|cp|$**, hence** $d(p,[ab]) \geqslant (a,b)_p$**.**

Step 2: Since $\triangle(abp)$ is δ-slim, there exists a point c' on $[ap]$ or $[bp]$ such that $d(c,c') \leqslant \delta$. Assume that $c' \in [pa]$. Using $|cp| = d(p,[ab])$, we obtain

$$2(c,a)_p = |ap|+|cp|-|ac| \leqslant 2\delta+|ap|+|c'p|-|ac'| = 2\delta+2|c'p| \leqslant 4\delta+2|cp| = 4\delta+2d(p,[ab]).$$

Step 3:

$$(a,b)_p = (a,c)_p + (b,c)_p - |pc| = (a,c)_p + \frac{1}{2}(|pb| - |pc| - |bc|) \leqslant (a,c)_p$$

(the last inequality follows from the triangle inequality). Applying the inequality from Step 2, obtain $(a,b)_p \geqslant d(p,[ab]) - 2\delta$. ■

COROLLARY: If X is Rips 0-hyperbolic, we have $d(p,[ab]) = (a,b)_p$.

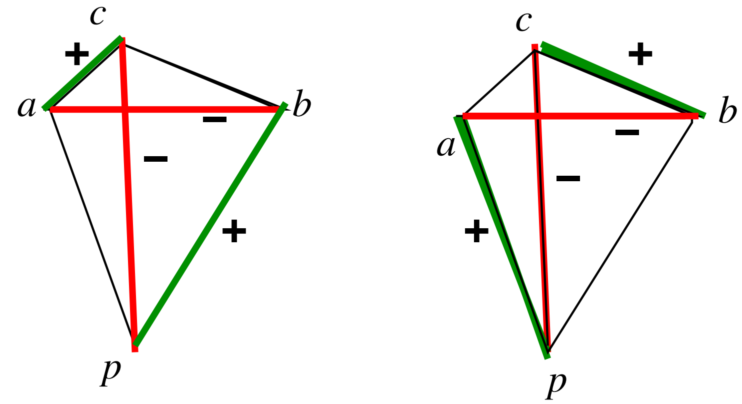
The Gromov inequality

DEFINITION: Let (X, p) be a metric space with a marked point, and $a, b, c \in X$. **The Gromov inequality**, or **the δ-Gromov inequality** is the inequality between the pairwise Gromov products,

$$(a, b)_p \geqslant \min\left[(a, c)_p, (b, c)_p\right] - \delta.$$

REMARK 1: The Gromov inequality is equivalent to the condition

$$\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta.$$



REMARK: The 0-Gromov inequality is $(a, b)_p \geqslant \min((a, c)_p, (b, c)_p)$; this means that **two smallest numbers in the triple $(a, b)_p, (a, c)_p, (b, c)_p$ are equal.**

The Gromov inequality: dependence on the choice of p

LEMMA: $(x,y)_p - (x,y)_t| \leqslant 2|pt|$.

Proof: Using triangle inequality in form $|xp| - |xt| \leqslant |pt|$ and $|yp| - |yt| \leqslant |pt|$, we obtain $(x,y)_p - (x,y)_t = \frac{1}{2}(|xp| + |yp| - |xt| - |yt|) \leqslant |pt|$. ■

THEOREM: Suppose that (X,p) satisfies the δ-Gromov inequality. Then for any $t \in X$, the space (X,t) **satisfies the $\delta + 2|pt|$-Gromov inequality.**

Proof: Applying the previous lemma to $(a,b)_p \geqslant \min[(a,c)_p, (b,c)_p] - \delta$, we obtain $(a,b)_t + 2|pt| \geqslant \min[(a,c)_t, (b,c)_t] - 2|pt| - \delta$. ■

The Gromov inequality: dependence on the choice of p (2)

A stronger result is true:

THEOREM: Suppose that (X, p) satisfies the δ-Gromov inequality. Then for any $t \in X$, the space (X, t) **satisfies the 2δ-Gromov inequality.**

Proof. Step 1: For all $a, b, c \in \mathbb{R}$, we have $\min(a, b) + \min(a, c) \geqslant \min(2a, b + c)$.

Step 2: Adding the Gromov inequalities for (t, y, z) and (z, x, y), and using Step 1, we obtain

$$(t, y)_p + (z, x)_p - \min[(t, z)_p + (x, y)_p, 2(y, z)_p] \geqslant -2\delta$$

for (t, x, y) and (z, x, t),

$$(t, y)_p + (z, x)_p - \min[(t, z)_p + (x, y)_p, 2(x, t)_p] \geqslant -2\delta.$$

Step 3: Applying Step 1 again, we see that half of the sum of these inequalities gives

$$(t, y)_p + (z, x)_p - \min[(t, z)_p + (x, y)_p, (y, z)_p + (x, t)_p] \geqslant -2\delta.$$

Step 4: The last inequality gives

$$-|ty| - |zx| + \max(|tz| + |zy|, |yz| + |xt|) \geqslant -2\delta$$

which is the same as $(t, y)_x - \min[(t, z)_x, (y, z)_x] \geqslant -2\delta$ (Remark 1). ■

Gromov hyperbolicity

DEFINITION: Let X be a metric space, not necessarily geodesic It is called **Gromov hyperbolic** if it satisfies the δ-Gromov inequality, for some $\delta \in \mathbb{R}$. The pair (X, p) is **δ-Gromov hyperbolic** if it satisfies the δ-Gromov inequality.

THEOREM: **Gromov hyperbolicity is equivalent to Rips hyperbolicity** (that is, hyperbolicity defined in terms of δ-slim triangles).

Proof: Next lecture

Eliyahu Rips (born 12 December 1948)

A mathematician has discovered a hidden code in The Bible that appears to reveal the details of events that have taken place thousands of years after The Bible was written, Eliyahu Rips disclosed in a letter to Yitzhak Rabin, the Prime Minister of Israel.

"The reason I'm telling you about this is that the only time your full name - Yitzhak Rabin - is encoded in The Bible, the words ¡¡assassin that will assassinate¿¿ cross your name.

That should not be ignored, because the assassinations of both John and Robert Kennedy and Anwar Sadat are also encoded in The Bible - in the case of Sadat with the first and last names of his killer, the date of the murder, the place, and how it was done.

I think you are in real danger, but that the danger can be averted."