Metric spaces

lecture 16: Quasi-isometries

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Quasi-isometries

DEFINITION: A map $f: X \longrightarrow Y$ is called **bi-Lipschitz with constant C**, or just **bi-Lipschitz**, if it is bijective, and both f and f^{-1} are C-Lipschitz (that is, satisfy $d(f(x), f(y)) \leqslant Cd(x, y)$). Two spaces X, Y are **bi-Lipschitz equivalent** if there exists a bi-Lipschitz map $f: X \longrightarrow Y$.

DEFINITION: The spaces X and Y are **quasi-isometric**, if X and Y are equipped with a ε-networks $X_\varepsilon \subset X$, $Y_\varepsilon \subset Y$ which are bi-Lipschitz equivalent.

DEFINITION: A map $f: X \longrightarrow Y$ of metric spaces is called **a quasi-metric map** if for some constants C, $\varepsilon > 0$, we have $d(f(x), f(y)) \leqslant Cd(x, y) + \delta$.

REMARK: A quasi-metric map is not necessarily continuous.

Quasi-isometries and quasi-metric maps

THEOREM: Let X, Y be metric spaces. **Then the following conditions are equivalent:**
(a) There exist quasimetric maps $f: X \longrightarrow Y$, $g: Y \longrightarrow X$, and a constant $A > 0$ such that $d(gf(x), x) < A$ **and** $d(fg(y), y) < A$ **for any** $x \in X, y \in Y$**.**
(b) The spaces X and Y **are quasi-isometric.**

Proof: (b) $\Rightarrow$ (a): Let N_X, N_Y be ε-nets in X, Y, and $\varphi: N_X \longrightarrow N_Y$ a bi-Lipschitz map. Choose projection maps $\Pi_X: X \longrightarrow N_X$, $\Pi_Y: Y \longrightarrow N_Y$ in such a way that every point is mapped to a point of an ε-net in its ε-neighbourhood. **Then** $f := \Pi_X \circ \varphi$ **and** $g := \Pi_Y \circ \varphi^{-1}$ **are quasi-isometries which satisfy** $gf = \Pi_X$ **and** $fg = \Pi_Y$**.**

The proof of (a) $\Rightarrow$ (b) follows later in this lecture.

COROLLARY: **Quasi-isometry is an equivalence relation.** ∎

Quasi-isometries and ε-nets

DEFINITION: An ε-net N in (X,d) is **δ-separated** if for any distinct $a,b \in N$, we have $d(a,b) > \delta$.

CLAIM: Let N be an ε-net in a metric space (X,d). Then **one there exists a ε-separated 2ε-net $N_1 \subset N$.**

Proof: Choose an order $\prec$ in N such that it is well-ordered, and remove any $x \in N$ if it belongs to a union of ε-balls with centers in $y \prec N$. Since $B_x(\varepsilon) \subset B_y(2\varepsilon)$, the new set $N \backslash \{x\}$ remains a 2ε-net. Repeating this process and using induction in a well ordered set N, we obtain a 2ε-net which is by construction ε-separated. ■

Lemma 1: Let $f: M \longrightarrow M'$ be a quasimetric map. Then **there exists $B > 0$ such that for any B-separated ε-net, the restriction $f\big|_{N_X}$ is Lipschitz.**

Proof: Let $d(f(x), f(y)) \leqslant Cd(x,y) + \delta$. Take $B > C\delta$, then $f\big|_{N_X}$ is $2C$-Lipschitz, because for all $x \neq y$, one has $d(x,y) > B$, giving $d(x,y) < 2d(x,y) - B$, hence

$$d(f(x), f(y)) \leqslant Cd(x,y) - \delta \leqslant 2Cd(x,y) - BC + \delta \leqslant 2Cd(x,y).$$

■

Quasi-isometries and quasi-metric maps (2)

THEOREM: Let X, Y be metric spaces. **Then the following conditions are equivalent:**
(a) There exist quasimetric maps $f: X \longrightarrow Y$, $g: Y \longrightarrow X$, and a constant $A > 0$ such that $d(gf(x), x) < A$ **and** $d(fg(y), y) < A$ **for any** $x \in X, y \in Y$**.**
(b) The spaces X and Y **are quasi-isometric.**

Proof of (a) $\Rightarrow$ (b). Step 1: Let $N_X \subset X$ be an ε-net. Since the $d(fg(Y), Y) < A$, the A-neighbourhood of the image of f contains Y. Since $d(f(x), f(y)) \leqslant Cd(x, y) + \delta$, $f(N_X)$ belongs to an $C\varepsilon + \delta$-neighbourhood of $f(X)$. **Therefore, the set $f(N_X)$ is an $C\varepsilon + \delta + A$-net.**

Step 2: Since $d(gf(x), x) < A$, and f and g are quasi-metric maps, we have $d(gf(x), gf(y)) \geqslant d(x, y) - d(x, gf(x)) - d(y, gf(y)) \geqslant d(x, y) - 2A$. On the other hand, $d(f(x), f(y)) \geqslant Cd(gf(x), gf(y)) - \delta$, because f is a quasi-metric map. Comparing these inequalities, we obtain

$$d(f(x), f(y)) \geqslant Cd(gf(x), gf(y)) - \delta \geqslant Cd(x, y) - 2CA - \delta.$$

This implies that $f(N_X)$ is $CR - 2CA - \delta$-separated if N is R-separated.

Quasi-isometries and quasi-metric maps (3)

THEOREM: Let X, Y be metric spaces. **Then the following conditions are equivalent:**
(a) There exist quasimetric maps $f: X \longrightarrow Y$, $g: Y \longrightarrow X$, and a constant $A > 0$ such that $d(gf(x), x) < A$ **and** $d(fg(y), y) < A$ **for any** $x \in X, y \in Y$**.**
(b) The spaces X and Y **are quasi-isometric.**

Step 1: For any ε-net N_X, **the set** $f(N_X)$ **is an** $C\varepsilon + \delta + A$**-net.**

Step 2: The set $f(N_X)$ **is** $CR - 2CA - \delta$**-separated if** N **is** R**-separated.**

Step 3: Take a R-separated $2R$-net in X. Step 2 implies that $f(N_X)$ is $CR - 2CA - \delta$-separated; choosing R sufficiently big, we obtain that $f\big|_{N_X}$ is bijective onto its image.

Choose B in such a way that f restricted to any B-separated net in X and g restricted to any B-separated net in Y is Lipschitz (Lemma 1). Take R such that $R \geqslant B$ and $CR - 2CA - \delta \geqslant B$, and let N_X be an R-separated $2B$-net. Then $N_Y := f(N_X)$ is a B-separated (Step 2) $CR + \delta + A$-net (Step 1), hence $g\big|_{N_Y}$ is also Lipschitz.

Step 4: It remains to show that $f^{-1}: N_Y \longrightarrow N_X$ is also Lipschitz; if not, there exists a sequence $t_i, z_i \in N_X$ such that $d(f(t_i), f(z_i)) \leqslant C_i d(t_i, z_i)$, and $\lim_i C_i = 0$. This would imply (using the same argument as in Step 1) that

$$d(t_i, z_i) - 2A \leqslant d(gf(t_i), gf(z_i)) \leqslant Cd(f(t_i), f(z_i)) + \delta \leqslant CC_i^{-1} d(f(t_i), f(z_i)) + \delta$$

This is impossible, because $d(f(t_i), f(z_i))$ is bounded from below (Step 2). **We have found ε-nets $N_X \subset X$ and $N_Y \subset Y$ such that $f: N_X \longrightarrow N_Y$ is bi-Lipschitz.** ■

Word metric on a group

DEFINITION: Let G be a group, S a collection of generators, and $\Gamma_{G,S}$ its Cayley graph. **The word metric** on G is defined as the restriction of the graph metric from $\Gamma_{G,S}$ to $G \subset \Gamma_{G,S}$.

REMARK: Let $\gamma \in G$ and suppose that the shortest decomposition $\gamma = \prod_i s_i$, where $s_i \in S$ are generators, has length r. Then $d_S(1, \gamma) = r$, "the length of the smallest word in letters $s_1, ..., s_n$ expressing γ". **This is why d_S is called "the word metric"**.

PROPOSITION: Let S, S' be a collection of generators, and $\max_{s \in S} d_{S'}(1, s) = C$. Denote by d_S, $d_{S'}$ the word metrics on G associated with S and S'. **Then the identity map $(\Gamma, d_S) \longrightarrow \Gamma, d_{S'})$ is C-Lipschitz**.

Proof: The word distance $r := d_S(g, h)$ is equal to the smallest length of a decomposition $g^{-1}h = \prod_i s_i$, where $s_i \in S$. **Replacing each s_i by its decomposition $s_i = \prod_i s_i'$, we obtain a word on the letters s_i' of length at most rC.** ■

COROLLARY: For any finitely generated group, all its Cayley graphs are quasi-isometric. ■

Gromov hyperbolicity is a quasi-isometry invariant

Later in this course I will prove the following theorem.

THEOREM: Let X, Y be quasi-isometric metric spaces. If X is Gromov hyperbolic, then Y is also Gromov hyperbolic.

This can be applied to Cayley graphs of a group.

COROLLARY: Assume that the Cayley graph of a finitely generated group is Gromov hyperbolic for one set generators. **Then it is Gromov hyperbolic for any generator set.** ■