Metric spaces

lecture 17: Approximation tree

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The Gromov product (reminder)

DEFINITION: Let $p \in X$ be a point in a metric space. **The Gromov product** $(a,b)_p$ is defined $(a,b)_p := 1/2(|ap| + |bp| - |ab|)$. **It measures for how long the geodesics from p to a and b stay together.**

REMARK: The distance function can be recovered from $(a,b)_p$. Indeed, $(a,a)_p = |ap|$, hence $|ab| = (a,a)_p + (b,b)_p - 2(a,b)_p$.

It is possible to define the distance in terms of the Gromov product.

DEFINITION: Let X be a set, and $p \in X$. We say that the function $(\cdot,\cdot)_p : X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ **satisfies the axiom of Gromov product** if the following conditions are satisfied:

[It is symmetric:] $(a,b)_p = (b,a)_p$.
[Non-degenerate:] $(a,a)_p = (a,b)_p = (b,b)_p \Leftrightarrow a = b$.
[Triangle inequality for Gromov product:] $(a,b)_p + (b,c)_p \leqslant (a,c)_p + (b,b)_p$.

CLAIM: Let $(a,b)_p$ is a function $X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ which satisfies the axioms of the Gromov product. **Then $d(a,b) := (a,a)_p + (b,b)_p - 2(a,b)_p$ is a metric on X.** Without the non-degeneracy, this formula defines a pseudometric. ■

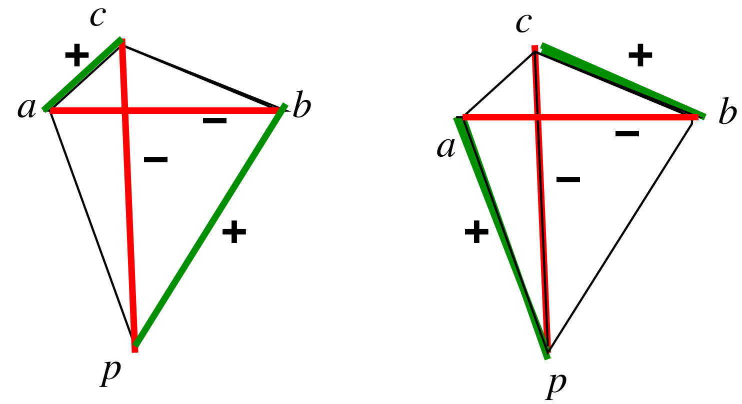
The Gromov inequality (reminder)

DEFINITION: Let (X,p) be a metric space with a marked point, and $a,b,c \in X$. **The Gromov inequality**, or **the δ-Gromov inequality** is the inequality between the pairwise Gromov products,

$$(a,b)_p \geqslant \min\left[(a,c)_p, (b,c)_p\right] - \delta.$$

REMARK: The Gromov inequality is equivalent to the condition

$$\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta.$$



REMARK 2: The 0-Gromov inequality is $(a,b)_p \geqslant \min((a,c)_p, (b,c)_p)$; this means that **two smallest numbers in the triple $(a,b)_p, (a,c)_p, (b,c)_p$ are equal.**

THEOREM: Suppose that (X,p) satisfies the δ-Gromov inequality. Then for any $t \in X$, the space (X,t) **satisfies the 2δ-Gromov inequality.**

A 0-hyperbolic space approximating a metric space

PROPOSITION: Let (X, p) be a metric space with a marked point. For any set $S = \{x = x_0, x_1, ..., x_n, x_{n+1} = y\} \subset X$, let $L_S(x, y) := \min_i (x_i, x_{i+1})_p$. Define the function $(x, y)'_p := \sup_S L_S(x, y)$, where the supremum is taken over all $x_1, ..., x_n \in X$. Let $d'(x, y) = d(x, p) + d(y, p) - 2(x, y)'_p$. **Then:**
(1) $d(x, y) \geqslant d'(x, y) \geqslant 0$
(2) $(x, y)'_p$ **satisfies the 0-Gromov inequality:** for any triple a, b, c, two of the numbers $(a, b)'_p, (a, c)'_p, (b, c)'_p$ are equal, and the third is smaller.
(3) d' **is a pseudo-metric.**

Proof. Step 1: (1) is clear. To prove (2), take the chain which connects a to b, and another which connects b to c. The union of these chains connects a to c. Therefore, $(a, c)'_p$, which is the supremum of $\min_i (x_i, x_{i+1})_p$ for all chains connecting a to c, is $\geqslant (a, b)'_p$ or $\geqslant (b, c)'_p$.

It remains to prove the triangle inequality for the function $d'(x, y) = d(x, p) + d(y, p) - 2(x, y)'_p$.

A 0-hyperbolic space approximating a metric space (2)

Step 1: It remains to prove the triangle inequality for Gromov product,

$$(a,b)'_p + (b,c)'_p \leqslant (a,c)'_p + (b,b)'_p. \qquad (*)$$

Step 2: 0-Gromov inequality for $(\cdot,\cdot)'_p$ (Remark 2), applied to the triple a,a,b, implies $(a,b)'_p \leqslant (b,b)'_p$; similarly, $(c,b)'_p \leqslant (b,b)'_p$.

Step 3: In the triple $(a,b)'_p, (a,c)'_p, (b,c)'_p$ two numbers are equal, and the third is $\geqslant$ than these two (Remark 2). If $(a,c)'_p = (b,c)'_p$ or $(a,c)'_p = (a,b)'_p$, then (*) follows from $(a,b)'_p \leqslant (b,b)'_p$ or $(b,c)'_p \leqslant (b,b)'_p$ (Step 2). If $(a,c)'_p$ is the biggest of the three, $(a,c)'_p \geqslant (b,c)'_p = (a,b)'_p$, then $(a,b)'_p \leqslant (b,b)'_p$ (Step 2), bringing

$$(a,c)'_p + (b,b)'_p \geqslant (a,c)'_p + (a,b)'_p \geqslant (a,b)'_p + (b,c)'_p. \quad \blacksquare$$

REMARK: For any $x \in X$, we have $(x,x)_p = (x,x)'_p$, because $(x,y)_p \leqslant (x,x)_p$, hence the supremum in $(x,x)'_p := \sup_S \min_i (x_i, x_{i+1})_p$ is reached when we take one term $(x,x)_p$.

The approximation tree

DEFINITION: Let X' be the Gromov 0-hyperbolic metric space, obtained from the pseudometric (X, d') gluing all pairs x, y with $d'(x, y) = 0$. Consider a tree X_{tr} with the set of vertices X', obtained as follows. For any $x \in X_{tr}$, we connect x to p by an interval of length $(x, x)'_p$, and glue the intervals $[a, p]$ and $[b, p]$ in a smaller interval of length $(a, b)'_p$ starting at p. It is called **the approximation tree for X.**

CLAIM: For any $x, y \in X' \subset X_{tr}$, the Gromov product of x and y in X_{tr} is equal to $(x, y)'_p$, and and the tautological map $(X, d) \xrightarrow{\nu} X' \subset X_{tr}$ is 1-Lipschitz.

Proof. Step 1: X_{tr} is a tree by construction; its Gromov product is equal to $(x, y)'_p$, because the Gromov product in a tree is a distance from p to $[x, y]$. Therefore, the image of X in X_{tr} is isometric to (X', d'). ■

Multi-Gromov inequality

We denote the minimum of $x, y, z, ... \in \mathbb{R}$ as $x \wedge y \wedge z \wedge ...$,

PROPOSITION 2: Suppose that a metric space X satisfies the following "multi-Gromov inequality": for any chain of points $x_1, ..., x_n$,

$$(x,y)_p \geqslant (x,x_1)_p \wedge (x_1,x_2)_p \wedge ... \wedge (x_n,y)_p - \delta'. \quad (*)$$

Let (X', d') be the space of vertices of its approximation tree. **Then the tautological map $(X,d) \xrightarrow{\nu} (X',d')$ has codiameter $\leqslant 2\delta'$**

Proof: Since $(x,y)'_p = \sup_{x_1,...,x_n} (x,x_1)_p \wedge (x_1,x_2)_p \wedge ... \wedge (x_n,y)_p$, (*) implies $(x,y)'_p \geqslant (x,y)_p \geqslant (x,y)'_p - \delta'$. By definition, $d'(x,y) = -2(x,y)'_p + 1/2|xp| + 1/2|yp|$ and $d(x,y) = -2(x,y)_p + 1/2|xp| + 1/2|yp|$. Then $(x,y)'_p \geqslant (x,y)_p \geqslant (x,y)'_p - \delta'$ gives $d'(x,y) + 2\delta' \geqslant d(x,y) \geqslant d'(x,y)$. ■

Multi-Gromov inequality for δ-Gromov hyperbolic spaces

PROPOSITION: Let (X, p) be a finite metric space satisfying the δ-gromov inequality which has $2^k + 1$ points. **Then X satisfies the multi-Gromov inequality for $\delta' = k\delta$.**

Proof: Suppose that for some numbers a, b, c, a_1, b_1, c_1 we have $a \geqslant b \wedge c - \delta$ and $a_1 \geqslant b_1 \wedge c_1 - \delta$. Then $a \wedge a_1 \geqslant b \wedge c \wedge b_1 \wedge c_1 - \delta$.

Step 2: Taking a Gromov hyperbolic space which has 5 points $x = x_0, y = x_1$ and $x_{1/4}, x_{1/2}, x_{3/4}$, and applying Step 1 to their pairwise Gromov products, we obtain that

$$(x_0, x_1)_p \geqslant (x_0, x_{1/2})_p \wedge (x_{1/2}, x_1)_p - \delta \geqslant \\ \geqslant (x_0, x_{1/4})_p \wedge (x_{1/4}, x_{1/2})_p \wedge (x_{1/2}, x_{3/4})_p \wedge (x_{3/4}, x_1)_p - 2\delta.$$

Multi-Gromov inequality for δ-Gromov hyperbolic spaces (2)

PROPOSITION: Let (X,p) be a finite metric space satisfying the δ-gromov inequality which has 2^k+1 points. **Then X satisfies the multi-Gromov inequality for $\delta' = k\delta$.**

Step 3: We prove the multi-Gromov inequality using induction in n. Suppose that we have $2^{n-1}-1$ points numbered by dyadic rationals $\frac{a}{2^{n-1}}$, $0<a<2^{n-1}$, and suppose (induction assumption)

$$(x_0,x_1)_p \geqslant (x_0,x_{1/2^{n-1}})_p \wedge (x_{1/2^{n-1}},x_{2/2^{n-1}})_p \wedge \ldots \wedge \left(x_{\frac{2^{n-1}-1}{2^{n-1}}},x_1\right)_p - (n-1)\delta. \quad (**)$$

Applying the inequality $\left(x_{\frac{m}{2^{n-1}}},x_{\frac{m+1}{2^{n-1}}}\right)_p \geqslant \left(x_{\frac{2m}{2^n}},x_{\frac{2m+1}{2^n}}\right)_p \wedge \left(x_{\frac{2m+1}{2^n}},x_{\frac{2m+2}{2^n}}\right)_p - \delta$ to each term and substituting it into (**), we obtain

$$(x_0,x_1)_p \geqslant (x_0,x_{1/2^{n-1}})_p \wedge (x_{1/2^{n-1}},x_{2/2^{n-1}})_p \wedge \ldots \wedge \left(x_{\frac{2^{n-1}-1}{2^{n-1}}},x_1\right)_p - (n-1)\delta \geqslant$$
$$\geqslant (x_0,x_{1/2^n})_p \wedge (x_{1/2^n},x_{2/2^n})_p \wedge \ldots \wedge \left(x_{\frac{2^n-1}{2^n}},x_1\right)_p - n\delta.$$

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