Metric spaces

lecture 18: Gromov hyperbolicity is equivalent to Rips hyperbolicity

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February 13, 2022, 17:00

The Gromov product (reminder)

DEFINITION: Let $p \in X$ be a point in a metric space. **The Gromov product** $(a,b)_p$ is defined $(a,b)_p := 1/2(|ap| + |bp| - |ab|)$. **It measures for how long the geodesics from p to a and b stay together.**

REMARK: The distance function can be recovered from $(a,b)_p$. Indeed, $(a,a)_p = |ap|$, hence $|ab| = (a,a)_p + (b,b)_p - 2(a,b)_p$.

It is possible to define the distance in terms of the Gromov product.

DEFINITION: Let X be a set, and $p \in X$. We say that the function $(\cdot,\cdot)_p : X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ **satisfies the axiom of Gromov product** if the following conditions are satisfied:

[It is symmetric:] $(a,b)_p = (b,a)_p$.
[Non-degenerate:] $(a,a)_p = (a,b)_p = (b,b)_p \ \Leftrightarrow a = b$.
[Triangle inequality for Gromov product:] $(a,b)_p + (b,c)_p \leqslant (a,c)_p + (b,b)_p$.

CLAIM: Let $(a,b)_p$ is a function $X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ which satisfies the axioms of the Gromov product. **Then $d(a,b) := (a,a)_p + (b,b)_p - 2(a,b)_p$ is a metric on X.** Without the non-degeneracy, this formula defines a pseudometric. ■

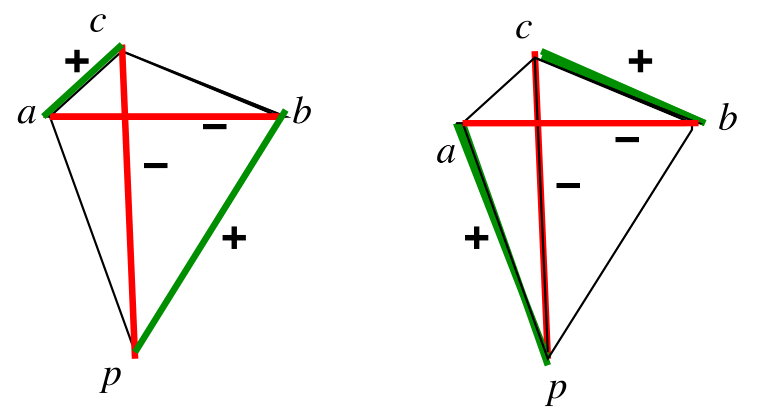
The Gromov inequality (reminder)

DEFINITION: Let (X, p) be a metric space with a marked point, and $a, b, c \in X$. **The Gromov inequality**, or **the δ-Gromov inequality** is the inequality between the pairwise Gromov products,

$$(a, b)_p \geqslant \min\left[(a, c)_p, (b, c)_p\right] - \delta.$$

REMARK: The Gromov inequality is equivalent to the condition

$$\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta.$$



REMARK 2: The 0-Gromov inequality is $(a, b)_p \geqslant \min((a, c)_p, (b, c)_p)$; this means that **two smallest numbers in the triple $(a, b)_p, (a, c)_p, (b, c)_p$ are equal.**

THEOREM: Suppose that (X, p) satisfies the δ-Gromov inequality. Then for any $t \in X$, the space (X, t) **satisfies the 2δ-Gromov inequality.**

The approximation tree (reminder)

PROPOSITION: Let (X, p) be a metric space with a marked point. For any set $S = \{x = x_0, x_1, ..., x_n, x_{n+1} = y\} \subset X$, let $L_S(x, y) := \min_i (x_i, x_{i+1})_p$. Define the function $(x, y)'_p := \sup_S L_S(x, y)$, where the supremum is taken over all $x_1, ..., x_n \in X$. Let $d'(x, y) = d(x, p) + d(y, p) - 2(x, y)'_p$. **Then:**

(1) $d(x, y) \geqslant d'(x, y) \geqslant 0$

(2) $(x, y)'_p$ **satisfies the 0-Gromov inequality:** for any triple a, b, c, two of the numbers $(a, b)'_p, (a, c)'_p, (b, c)'_p$ are equal, and the third is smaller.

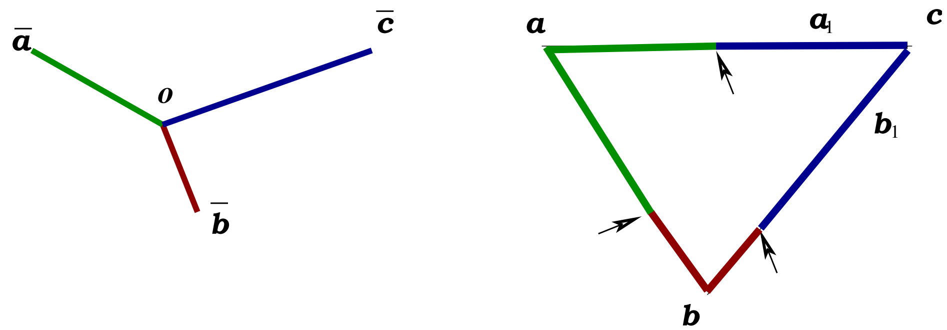
(3) d' **is a pseudo-metric.**

DEFINITION: Let X' be the Gromov 0-hyperbolic metric space, obtained from the pseudometric (X, d') gluing all pairs x, y with $d'(x, y) = 0$. Consider a tree X_{tr} with the set of vertices X', obtained as follows. For any $x \in X_{tr}$, we connect x to p by an interval of length $(x, x)'_p$, and glue the intervals $[a, p]$ and $[b, p]$ in a smaller interval of length $(a, b)'_p$ starting at p. It is called **the approximation tree for X.**

PROPOSITION: Let (X, p) be a finite metric space satisfying the δ-gromov inequality which has $2^k + 1$ points. **Then the codiameter of the approximation map $\nu : X \longrightarrow X_{tr}$ satisfies** $\operatorname{codiam}' nu \leqslant k\delta$**.**

Approximation tripod and δ-slim triangles (reminder)

DEFINITION: Let $\triangle(abc)$ be a geodesic triangle. Define **a model 0-hyperbolic triangle**, or **a model tripod** as a tree $\triangle(\overline{a}\overline{b}\overline{c})$ with three free ends



and three edges, connected in a fourth vertex o, such that the corresponding distances are equal: $|ab| = |\overline{a}\overline{b}|$, $|ac| = |\overline{a}\overline{c}|$, $|bc| = |\overline{b}\overline{c}|$; this also gives $|o,\overline{a}| = (b,c)_a$, $|o,\overline{b}| = (a,c)_a$, $|o,\overline{c}| = (a,b)_a$.

PROPOSITION: Let $\Psi : \triangle(abc) \longrightarrow \triangle(\overline{a}\overline{b}\overline{c})$ be the comparison map to the model tripod defined above. Then

(a) **If** $\operatorname{codiam}\Psi \leqslant \delta$, **the triangle** $\triangle(abc)$ **is** δ**-slim.**

(b) **If** $\triangle(abc)$ **is** δ**-slim, then** $\operatorname{codiam}\Psi \leqslant 2\delta$**.**

Multi-Gromov inequality and δ-slim triangles

PROPOSITION 4: Let (X,c) be a metric space with geodesic metric, satisfying the δ-Gromov inequality, and a_1, b_1 points on the sides $[ac]$, $[bc]$ of a geodesic triangle $\triangle(cab)$. Let Y be the 5 point metric space $Y = \{c, a, b, a_1, b_1\}$, and let $\nu: (Y,d) \longrightarrow (Y',d')$ be the approximation map. **Then** $\operatorname{codiam} \nu \leqslant 2\delta$**.**

Proof: Codiameter of the approximation tree map $\nu: X \longrightarrow X_{tr}$ is $k\delta$, if Y is a finite set, $|Y| \leqslant 2^k + 1$. **Therefore,** $\operatorname{codiam} \nu \leqslant 2\delta$**.** ■

THEOREM: Suppose that a geodesic space (X,p) satisfies the δ-Gromov inequality. **Then** X **is Rips** 12δ**-hyperbolic**, that is, all geodesic triangles in X are 12δ-slim.

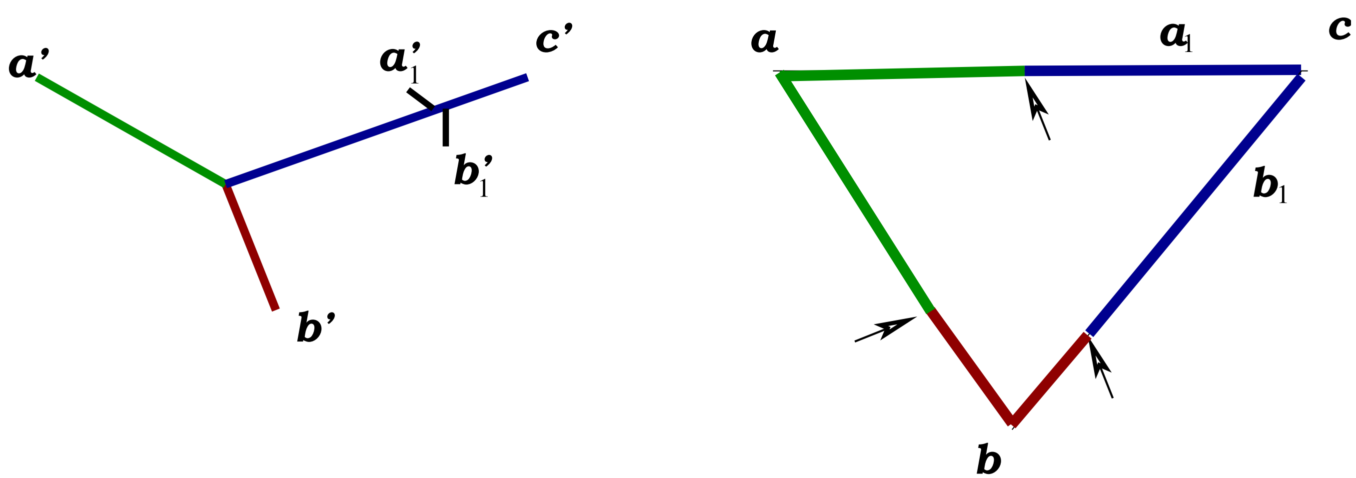
REMARK: **It is possible to show that X is Rips 6δ-hyperbolic, instead of 12δ.**

REMARK: Its proof goes as follows: consider the map $\mu: \triangle(abc) \longrightarrow T$ to the model tripod, and let a_1, b_1 be two preimages of the same point. It suffices to show to show that $d(a_1, b_1) \leqslant 12\delta$. To prove this, we take the approximation tree Y_{tr} associated with the set $Y := \{c, a, b, a_1, b_1\}$, and show that the distance between the images of a_1 and b_1 in Y_{tr} is $\leqslant 8\delta$.

Multi-Gromov inequality and δ-slim triangles (2)

THEOREM: Suppose that a geodesic space (X, p) satisfies the δ-Gromov inequality. **Then X is Rips 12δ-hyperbolic**, that is, all geodesic triangles in X are 12δ-slim.

Proof. Step 1: The space (X, c) is 2δ-hyperbolic (Lecture 15, page 7). Consider the space $Y := \{c, a, b, a_1, b_1\}$, as above, and let Y_{tr} be its approximation tree.



Step 2: Let $\mu : \triangle(abc) \longrightarrow T$ be the map to the model tripod T. Choose the points $a_1 \in [a, c]$ and $b_1 \in [b, c]$ which are mapped to the same point of T; **to prove the theorem, we need to show that $d(a_1, b_1) \leqslant 12\delta$.** To do this, we show that the distance between $a_1' = \nu(a_1)$, $b_1' = \nu(b_1)$ is $\leqslant 8\delta$, and use $\operatorname{codiam} \nu \leqslant 4\delta$ (Step 1).

Multi-Gromov inequality and δ-slim triangles (2)

Step 3: Let $a_1 \in [a,c]$. Then $(a_1,a)'_c \geqslant d(a_1,c)'$ because $L_S(a_1,a)_c = d(a_1,c)$ when $S=\emptyset$, and $(a_1,y)_c \leqslant d(a_1,c)$ by triangle inequality, for all $y \in Y$. **We have shown that $(a_1,a)'_c = d(a_1,c)$.**

Step 4: Remark 2 implies that in the triple $(a_1,a)'_c$, $(a_1,b_1)'_c$, $(a,b_1)'_c$, two smallest terms are equal. Step 3 gives that $(a_1,a)'_c = d'(a_1,c) = d(a_1,c) = d(b_1,c)$, and $(a,b_1)'_c \leqslant d'(b_1,c) = d(b_1,c) = (a_1,a)'_c$, and also $(a_1,b_1)'_c \leqslant d'(a_1,c) = d(a_1,c) = (a_1,a)'_c$. **This implies $(a_1,b_1)'_c = (a,b_1)'_c = (b,a_1)'_c$.**

Step 5: By construction, $d(a_1,c) = d(b_1,c) \leqslant (a,b)_c$, hence $(a,b)_c \geqslant (a_1,a)_c = d(a_1,c)$; the inequality $(b,a_1)_c \leqslant (a_1,a)_c = d(a_1,c)$ follows from the triangle inequality. Step 3 implies that $d'(c,a_1) = (a,a_1)'_c \geqslant (b,a_1)'_c$. Then Remark 2 implies that either $(a_1,a)'_c = (a_1,b)'_c \leqslant (a,b)'_c$ or $(a_1,a)'_c \geqslant (a_1,b)'_c = (a,b)'_c$. In the latter case, $(a_1,a)'_c - (a_1,b)'_c \leqslant 4\delta$, because $(a_1,a)_c \leqslant (a,b)_c$, and $\operatorname{codiam} \nu \leqslant 4\delta$. Combining this with Step 4, **we obtain that either $(a_1,a)'_c = (a_1,b)'_c = (a_1,b_1)'_c$, or $|(a_1,a)'_c - (a_1,b_1)'_c| \leqslant 4\delta$.**

Step 6: $d'(a_1,b_1) = d'(a_1,c) + d'(b_1,c) - 2(a_1,b_1)'_c \leqslant 8\delta$ (Step 5). Then $d(a_1,b_1) \leqslant 12\delta$, because $\operatorname{codiam} \nu \leqslant 4\delta$. ∎

Gromov product and the distance to the geodesic (reminder).

PROPOSITION 1: Let $\triangle(abp)$ be a δ-slim triangle. **Then** $d(p, [ab]) \geqslant (a, b)_p \geqslant d(p, [ab]) - 2\delta$.

Proof. Step 1: Let c be the point of $[ab]$ closest to p. The triangle inequality gives $|ap| - |cp| + |bp| - |cp| \leqslant |ac| + |cb| = |ab|$. **This implies** $|ap| + |bp| - |ab| \leqslant 2|cp|$**, hence** $d(p, [ab]) \geqslant (a, b)_p$**.**

Step 2: Since $\triangle(abp)$ is δ-slim, there exists a point c' on $[ap]$ or $[bp]$ such that $d(c, c') \leqslant \delta$. Assume that $c' \in [pa]$. Using $|cp| = d(p, [ab])$, we obtain

$$2(c, a)_p = |ap| + |cp| - |ac| \leqslant 2\delta + |ap| + |c'p| - |ac'| =$$
$$= 2\delta + 2|c'p| \leqslant 4\delta + 2|cp| = 4\delta + 2d(p, [ab]).$$

Step 3:

$$(a, b)_p = (a, c)_p + (b, c)_p - |pc| = (a, c)_p + \frac{1}{2}(|pb| - |pc| - |bc|) \leqslant (a, c)_p$$

(the last inequality follows from the triangle inequality). Applying the inequality from Step 2, obtain $(a, b)_p \geqslant d(p, [ab]) - 2\delta$. ■

Gromov inequality for δ-slim triangles

LEMMA: Let $\triangle(abc)$ be a δ-slim triangle in a geodesic metric space, and p' a point on the side $[a,b]$ closest to p. Assume that p' belongs to a δ-neighbourhood of $[ac]$. **Then $(a,b)_p \geqslant (b,c)_p - 3\delta$.**

Proof. Step 1: Since $d(p',[ac]) \leqslant \delta$, we have $d(p,[a,b]) - d(p,[a,c]) \geqslant -\delta$ by triangle inequality.

Step 2: Proposition 1 implies that $(a,b)_p \leqslant d(p,[ab]) \leqslant (a,b)_p + 2\delta$. **Then Step 1 implies $(a,b)_p - (b,c)_p \geqslant -3\delta$.** ■

COROLLARY: Let X be a Rips δ-hyperbolic geodesic space. **Then X satisfies the 3δ-Gromov inequality.**

Proof: By the previous lemma, we have $(a,b)_p - (b,c)_p \geqslant -3\delta$ if $p' \in [ac](\delta)$ or $(a,b)_p - (a,c)_p \geqslant -3\delta$ if $p' \in [bc](\delta)$, Comparing these inequalities, we obtain that $(a,b)_p \geqslant \min((b,c)_p,(a,c)_p) - 3\delta$. ■

COROLLARY: A Rips δ-hyperbolic space is Gromov 3δ-hyperbolic. A Gromov δ-hyperbolic space is Rips 12δ-hyperbolic. ■

Strong and weak topology on the space of maps

DEFINITION: Let X, Y be metric spaces, and Map(X,Y) the set of all maps. Given a point $x \in X$ and am open set $W \subset Y$, consider the set $U_{x,W} = \{f \in \operatorname{Map}(X,Y) \mid f(x) \subset W\}$. **Topology of pointwise convergence**, or **weak topology**, or **Tychonoff topology** on Map(X,Y) is defined by a subbase of open sets $U_{x,W}$, $x \in X, W \subset Y$. **Topology of uniform convergence**, or **strong topology**, or **C^0-topology** is defined by the sub-base of sets $U_{f,\delta}$, where $f \in \operatorname{Map}(X,Y)$, $\delta > 0$, and

$$U_{f,\delta} := \{g \in \operatorname{Map}(X,Y), \quad \forall x \in X, d(f(x), g(x)) < \delta\}.$$

CLAIM: A sequence $\{f_i\} \subset \operatorname{Map}(X,Y)$ converges to f in C^0 if and only if $\lim_i \sup_{x\in X} d(f_i(x), f(x)) = 0$, and converges to f in Tychonoff $\Leftrightarrow$ for each $x \in X$ we have $\lim_i f_i(x) = f(x)$. ■

CLAIM: In C^0, a limit of a sequence of continuous maps is continuous, a limit of C-Lipschitz maps is C-Lipschitz.

REMARK: In Tychonoff topology, **a limit of a sequence of continuous maps is not necessarily continuous.** However, **a pointwise limit of C-Lipschitz maps is C-Lipschitz**.

Tychonoff theorem

PROPOSITION: Let X be countable, and Y compact and metrizable. **Then the space** $\mathrm{Map}(X,Y)$ **is sequentially compact in Tychonoff topology.**

Proof: We identify X and the set $\mathbb{N}$ of natural numbers. Then the elements $\mathrm{Map}(X,Y)$ are sequences of elements of Y. Consider a sequence of sequences $\{y_i(n)\}$; we need to find a subsequence which converges to $\{y(n)\}$. We take a subsequence for which $\lim_n y_1(n)$ converges, take its first element for $y(1)$. Then we choose a subsequence of this subsequence such that $\lim_n y_2(n)$ converges, and take its second element for $y(2)$, and so on. ■

REMARK: Tychonoff theorem **is true without these assumptions,** but it's proof is less constructive.

DEFINITION: A metric d on M is called **bounded** if (M,d) has finite diameter, and **separable** if it contains a dense, countable subset.

REMARK: The space $\mathrm{Map}(X,Y)$ with Tychonoff topology **is metrizable, if** X **is countable.** To see that, consider a metric d on Y; if d is non-bounded, replace it by a bounded metric which defines the same topology, such as $d_1(x,y) := \min(d(x,y),1)$. Then **the metric** $d(\{x(n)\},\{y(n)\}) := \sum \frac{d(x(i),y(i))}{2^n}$ **defines the Tychonoff topology on the set of sequences.**

Arzelà-Ascoli theorem

LEMMA: Let $X_0 \subset X$ be a dense, countable subset, X and Y compact, and $\{f_i \in \operatorname{Map}(X,Y)\}$ a sequence of C-Lipschitz maps. Assume that $\{f_i|_{X_0}\}$ converges in Tychonoff topology (that is, pointwise). **Then $\{f_i\}$ converges in C^0-topology, and the limit $f := \lim_i f_i$ is also C-Lipschitz.**

Proof. Step 1: Choose $a, b \in X_0$. Clearly, $\lim d(f_i(a), f(a)) = 0$ and $\lim d(f_i(b), f(b)) = 0$, hence $d(f(a), f(b)) = \lim_i d(f_i(a), f_i(b)) \leqslant Cd(a,b)$. Therefore, f is C-Lipschitz, hence we can extend it to the completion of X_0, and $\{f_i\}$ converges in Tychonoff to a limit f, which is also C-Lipschitz.

Step 2: Then f can be extended to a completion X (this is clear, because Lipschitz maps take Cauchy sequences to Cauchy sequences), and, moreover, f is a pointwise limit of f_i (left as an exercise).

Step 3: It remains to show that $\{f_i\}$ converges to f in C^0. Otherwise $\sup_{x\in X} d(f_i(x), f(x)) > A$ for infinitely many i; passing to a subsequence, we can assume that $\sup_{x\in X} d(f_i(x), f(x)) > A$ for all i. Let $\{x_i\}$ be a sequence of points such that $d(f_i(x_i), f(x_i)) > A$ for all i, and x its limit. Then

$$A < d(f_i(x_i), d(f(x_i)) \leqslant d(f_i(x_i), d(f_i(x)) + d(f_i(x), f(x)) + d(f(x), f(x_i)) \leqslant$$
$$\leqslant Cd(x_i, x) + d(f_i(x), f(x)) + Cd(x, x_i),$$

which is impossible, because $\lim_i d(x_i, x) = 0$ and $\lim_i d(f_i(x), f(x)) = 0$. ∎

Arzelà-Ascoli theorem (2)

LEMMA: Let $X_0 \subset X$ be a dense, countable subset, X and Y compact, and $\{f_i \in \operatorname{Map}(X,Y)\}$ a sequence of C-Lipschitz maps. Assume that $\{f_i\big|_{X_0}\}$ converges in Tychonoff topology (that is, pointwise). **Then $\{f_i\}$ converges in C^0-topology, and the limit $f := \lim_i f_i$ is also C-Lipschitz.**

COROLLARY: (Arzelà-Ascoli theorem for Lipschitz maps)
Let X be a separable, compact metric space, Y compact, and $L_C(X,Y) \subset \operatorname{Map}(X,Y)$ the space of C-Lipschitz maps. **Then $L_C(X,Y)$ is compact in uniform topology.**

Proof: Immediately follows from the above lemma and the Tychonoff theorem. ■

Rudolf Lipschitz

Rudolf Otto Sigismund Lipschitz
(1832 – 1903)