Metric spaces

lecture 19: Quasigeodesics

Misha Verbitsky

IMPA, sala 236

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Quasigeodesic metrics on metric graphs

DEFINITION: Let $\Gamma, |\cdot|$ be a finite metric graph. **C-quasigeodesic metric** on Γ is a metric d which satisfies $C^{-1}|x-y| \leqslant d(x,y) \leqslant |x-y|$ on each edge of the graph.

REMARK: A metric d on a finite metric graph $\Gamma, |\cdot|$ is quasigeodesic if and only if the identity map $(\Gamma, d) \longrightarrow (\Gamma, |\cdot|)$ is 1-Lipschitz, and the identity map $(\Gamma, |\cdot|) \longrightarrow (\Gamma, d)$ is C-Lipschitz.

CLAIM: Consider the space of metrics on Γ as a metric space with metric $d(d_1, d_2) := \sup_{(x,y)\in\Gamma^2} |d_1(x,y) - d_2(x,y)|$. **Then a limit of C-quasigeodesic metrics is C-quasigeodesic.**

Proof: $d(x,y) = \lim_i d_i(x,y) \leqslant |x,y|$, hence a limit of metrics d_i for which the identity map $(\Gamma, |\cdot|) \longrightarrow (\Gamma, d_i)$ and $(\Gamma, d) \longrightarrow (\Gamma, |\cdot|)$ is Lipschitz also satisfies this property. ■

Quasigeodesic metrics on metric graphs (2)

COROLLARY: The space of C-quasigeodesic metrics on $(\Gamma, |\cdot|)$ is compact with respect to the uniform convergence of $|\cdot|$ considered as a function $\Gamma\times\Gamma \longrightarrow \mathbb{R}^{\geqslant 0}$

Proof: The proof is more or less the same as for the Arzelà-Ascoli theorem. Let $\{d_i\}$ be a sequence of C-quasigeodesic metrics. Find a countable dense subset in $\Gamma_0 \subset \Gamma$ and replace $\{d_i\}$ by a subsequence which converges on Γ_0. Extend the limit metric from Γ_0 to its completion, using the Lipschitz property, and use the inequalities $C^{-1}|x-y| \leqslant d(x,y) \leqslant |x-y|$ to prove that d is C-quasigeodesic. ■

Quasigeodesics and Morse lemma

DEFINITION: A C-quasigeodesic in a metric space M is a rectifiable path $\gamma: [0,a] \longrightarrow M$ which satisfies $L(\gamma\big|_{[x,y]}) \leqslant Cd(x,y)$, where $L(\gamma\big|_{[x,y]})$ denotes the length of the interval of γ.

REMARK: I will tacitly consider that all quasigeodesics are naturally paramerized, that is, $L(\gamma\big|_{[x,y]}) = |x-y|$.

REMARK: Morse Lemma (in the original version of Morse) is a statement about the geometry of the hyperbolic plane $\mathbb{H}^2$: **for each $C > 1$ there is R such that any C-quasigeodesic connecting a to $b \in \mathbb{H}^2$ is contained in an R-neighbourhood of $[a,b]$.**

REMARK: This statement **is clearly false in $\mathbb{R}^2$ with the Euclidean metric.**

DEFINITION: Let γ be a finite C-quasigeodesic in a geodesic Gromov hyperbolic space, and $R(\gamma)$ be the maximum of the distance from the points on γ to any of the minimal geodesics connecting the ends of γ. Morse lemma claims that $R(\gamma)$ **is bounded by a constant, which depends only on M and C, for any C-quasigeodesic $\gamma \subset M$.**

Harold Calvin Marston Morse
(24 March 1892 - 22 June 1977)

Marston Morse and colleague at the dedication of the Institute for Advanced Study at Princeton, 1938.

Quasigeodesics in δ-hyperbolic spaces

CLAIM: Let (M, d_M) be a metric space, and $\gamma : [0, a] \longrightarrow M$ be a C-quasigeodesic connecting a to $b \in M$; we use the normal parametrization on γ. **Then the metric $d(x,y) := \frac{d_M(\gamma(ax),\gamma(ay))}{a}$ is C-quasigeodesic on $[0,1]$.** If, in addition, M is δ-hyperbolic, then **the space $([0,1], d)$ is δ/a-hyperbolic.**

Proof. Step 1: Then $C^{-1}|ax - ay| \leqslant d_M(ax, ay) \leqslant |ax - ay|$ because γ is C-quasigeodesic, which implies that $C^{-1}|x - y| \leqslant d(x,y) \leqslant |x - y|$, hence the metric d is C-quasigeodesic on $[0,1]$.

Step 2: The δ-Gromov inequality on $(\operatorname{im}\gamma, d_M)$ implies the δ/a-Gromov inequality in $([0,1], d)$ ■

REMARK: The metric d on $[0,1]$ **is not intrinsic.**

Limits of quasigeodesic metrics

DEFINITION: Let $\gamma_i : [0, a_i] \longrightarrow M$ be a sequence of C-quasigeodesics, with **A limit metric** is a limit (any of the limits) of the sequence of the C-geodesic metrics $d_i(x,y) := \frac{|\gamma_i(a_i x), \gamma_i(a_i y)|}{a_i}$ on $[0,1]$.

REMARK: The limit of d_i always exists, because the metric d_i is C-quasigeodesic, and the space of C-quasigeodesic metrics on $[0,1]$ is compact.

CLAIM: Let $\gamma_i : [0, a_i] \longrightarrow M$, $\lim_i a_i = \infty$, be a sequence of C-quasigeodesics in a δ-hyperbolic space, and $([0,1], d)$ is its limit metric. **Then the space $([0,1], d)$ is 0-hyperbolic**.

Proof: Let $(x,y)^i_p$ be the Gromov product on $[0,1]$ in d_i. Then $(x,y)^i_p \geqslant (x,z)^i_p \wedge (z,y)^i_p - \frac{\delta}{a_i}$ because d_i is $\frac{\delta}{a_i}$-hyperbolic. Passing to a limit, we obtain

$$\lim_i (x,y)^i_p \geqslant \lim_i (x,z)^i_p \wedge (z,y)^i_p - \frac{\delta}{a_i} = \lim_i (x,z)^i_p \wedge (z,y)^i_p. \blacksquare$$

Morse lemma: the case of digon

PROPOSITION: Let $\gamma_i : [0, a_i] \longrightarrow M$ be a sequence of C-quasigeodesics in a δ-hyperbolic geodesic space, with $\lim_i a_i = \infty$. Denote by X_i the union of the image of γ_i and a geodesic interval connecting its ends. Consider the metric d_i on the digon graph $\Diamond$ (two vertices, two edges), $d_i = \frac{d_M\big|_{X_i}}{a_i}$. **Then d_i contains a subsequence which uniformly converges to a 0-hyperbolic pseudo-metric $\tilde{d}$ on $\Diamond$.**

Proof: The space of C-quasigeodesic metrics on $\Diamond$ is compact, hence $\{d_i\}$ has a converging subsequence. The limit is Gromov 0-hyperbolic because $\lim_i \frac{\delta}{a_i} = 0$. ■

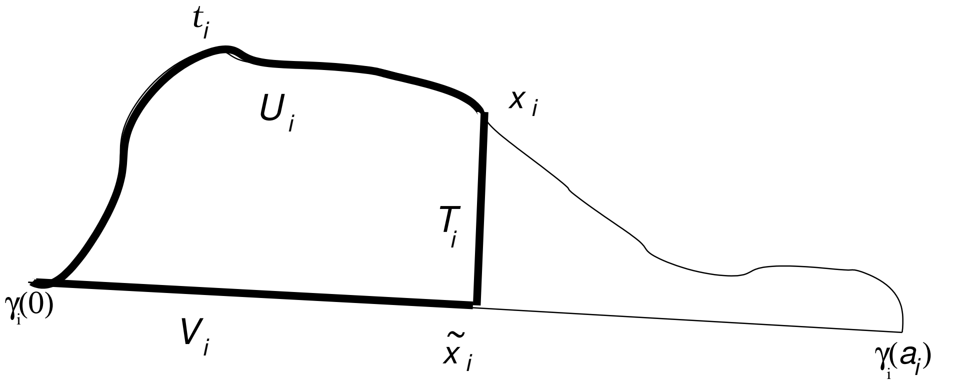
COROLLARY: Let $\{\gamma_i\}$ be a sequence of C-quasigeodesics in a geodesic δ-hyperbolic space, with the ends x_i and y_i. Fix a minimizing geodesic $[x_i, y_i]$. Assume that $\lim_i d(x_i, y_i) = \infty$. Denote by $R(\gamma_i)$ the maximal distance between a point of γ_i and $[x_i, y_i]$. **Then $\lim_i \frac{R(\gamma_i)}{a_i} = 0$.**

Proof: Consider the pseudometric digon $(\Diamond, d)$ obtained as the limit of digons $[x_i, y_i] \cup \gamma_i$. By construction, the distance between two sides of $(\Diamond, d)$ is equal to $\lim_i \frac{R(\gamma_i)}{a_i}$. However, $(\Diamond, d)$ is a tree, hence **this pseudometric digon is isometric to an interval, and the distance between its sides is 0.** ■

Morse lemma: the case of a triangle

Let $\gamma_i : [0, a_i] \longrightarrow M$ be a sequence of quasigeodesics in a hyperbolic space, such that $\lim_i a_i = \infty$ and $\lim_i R(\gamma_i) = \infty$, but $R(\gamma_i) \leqslant \frac{a_i}{2C}$. Denote by $t_i \in [0, a_i]$ the point such that the distance $d(\gamma(t_i), [\gamma(0), \gamma(a_i)])$ reaches maximum in t_i.

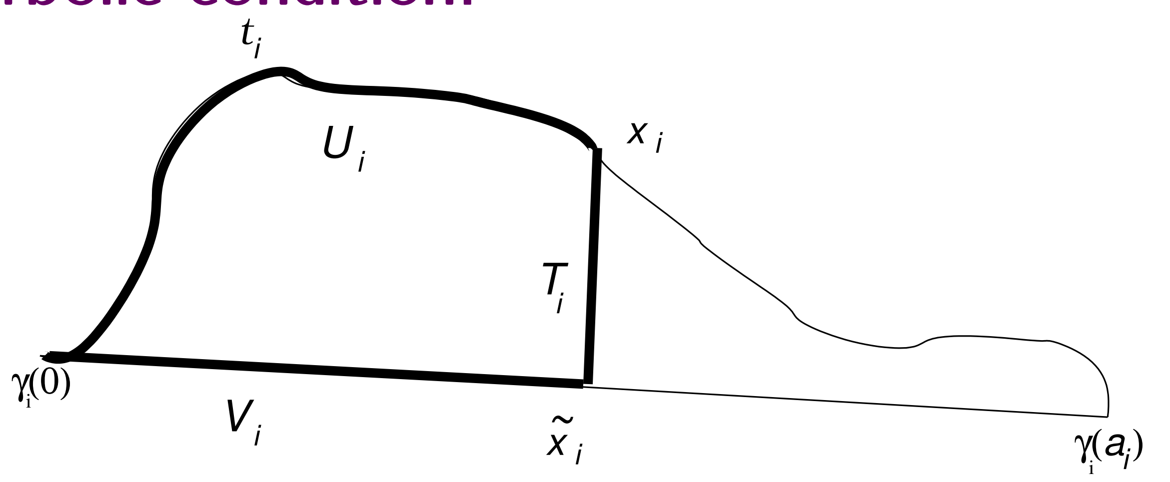
Assume that $d(t_i, \gamma_i(0)) \leqslant 2CR(\gamma_i)$ for all i, and take $x_i \in \gamma_i([0, a_i])$ such that $d(x_i, \gamma(0)) = 4CR(\gamma_i)$. Let x_i' be the point of $[\gamma(0), \gamma(a_i)]$ which is closest to x_i. Consider the curved triangle $Y_i \subset M$; one of its sides, denoted U_i, is an interval of γ_i from $\gamma_i(0)$ to $\gamma(x_i)$, another, denoted V_i, is a geodesic segment conecting x_i' to $\gamma_i(0)$, and the third side, denoted T_i is a geodesic interval connection x_i' to x_i.



The curved triangle Y_i is naturally identified with the graph $\triangle$ (3 vertices, 3 sides, connected successively). Denote by d_i the metric induced on Y_i by $\frac{d_M}{R(\gamma_i)}$.

Morse lemma: the case of a triangle (2)

The side U_i in $(\triangle, d_i)$ is a C-quasigeodesic, distance between its ends is $4C$, the adjacent sides V_i and T_i are geodesics with $|V_i| \leqslant 1$, $|T_i| \leqslant 4C + 1$. Therefore, **each d_i is a C-quasigeodesic on a graph Y_i, and a subsequence of $\{d_i\}$ converges uniformly to a pseudometric $\tilde{d}$ on $\triangle$, and $\tilde{d}$ satisfies the Gromov 0-hyperbolic condition.**



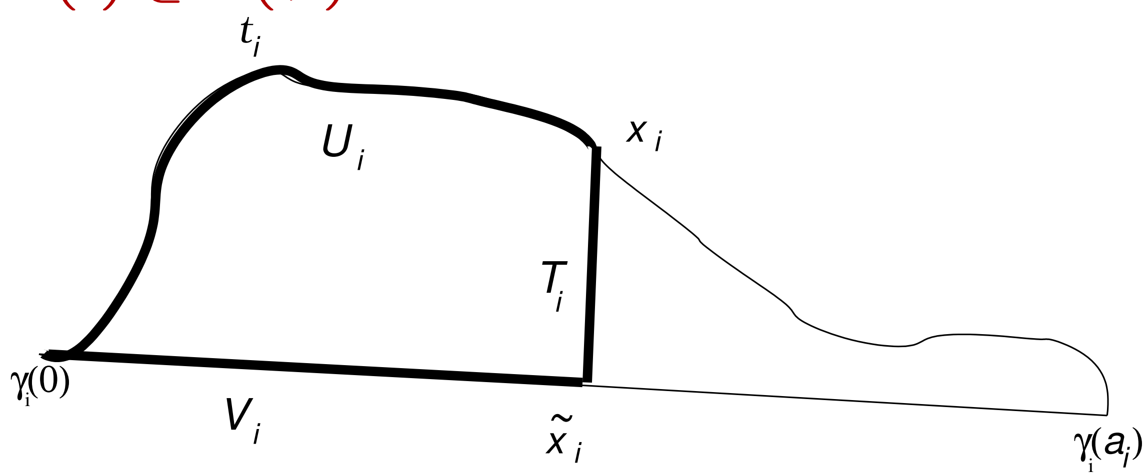
Let U be the limit of the side U_i, and V, T the limits of V_i, T_i. Then $|U| = 4C$. $|V| \leqslant 4C + 1$, $|T| \leqslant 1$.

Let $t \in U$ be the limit of $t_i \in U_i$. Since $d(\gamma(0), t_i) \leqslant 2CR(\gamma_i)$, the distance from t to the corresponding end of U is $\leqslant 2C$.

Morse lemma: the case of a triangle (finale)

Denote by $\mathbb{T}$ the metric space associated with the pseudometric d on the triangle $U \cup V \cup T$, and let Ψ be the tautological map.

PROPOSITION: Let $\Psi : \triangle \longrightarrow \mathbb{T}$ be a continuous map of a triangle with sides U, V, T to a tree, inducing isometry on V and T. Assume that U is mapped to an interval of length $2C$, and T to an interval of length $\leqslant 1$. Let $t \in U$ be a point of U satisfying $d(t, u) \leqslant 2C$, where u is the vertex connecting U and V. **Then $\Psi(t) \subset \Psi(V)$.**

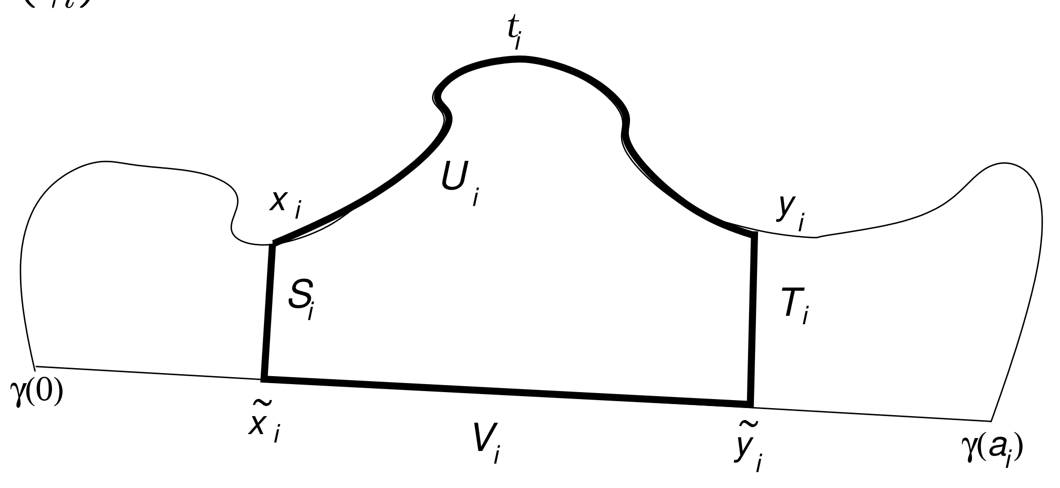


■

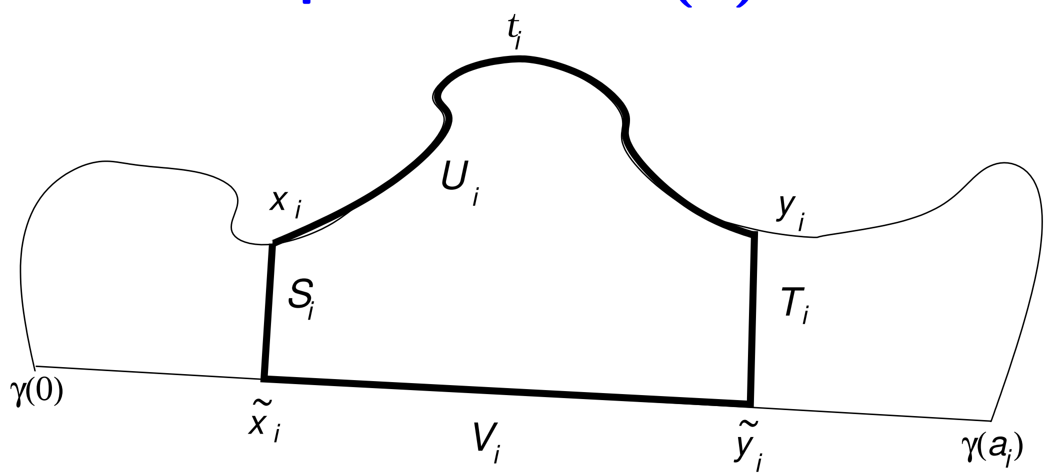
This implies $d(t, V) = 0$. However, by construction, $d_i(t_i, V_i) = 1$, which leads to a contradiction. **This implies that $\lim_i \frac{R(\gamma_i)}{a_i} = 0$ together with $R(\gamma_i) < \frac{a_i}{2C}$ implies $\lim_i R(\gamma_i) < R < \infty$.**

Morse lemma: the curved quadrilateral

Let $\gamma_i: [0,a_i] \longrightarrow M$ be a sequence of C-quasigeodesics in a hyperbolic space, such that $\lim_i a_i = \infty$ and $\lim_i R(\gamma_i) = \infty$, but $R(\gamma_i) < \frac{a_i}{2C}$. Let $t_i \in [0,a_i]$ be the point where the distance between $\gamma(t_i) and [\gamma(0), \gamma(a_i)]$ reaches its maximum. Take points x_i, y_i on $\gamma_i([0,a_i])$ such that $d(x_i,y_i) = 4CR(\gamma_i)$, and t_i lies in the middle of the interval of γ_i connecting x_i and y_i. Let Π_i be a curved quadrilateral, with one curved side, identified with the interval of γ_i from x_i to y_i, and other three sides geodesic intervals $[x_i, \tilde x_i]$, $[\tilde x_i, \tilde y_i]$, $[\tilde y_i, y_i]$, where $\tilde x_i, \tilde y_i$ are points of the minimizing geodesic $[\gamma_i(0), \gamma_i(a_i)]$ closest to x_i, y_i.
The quadrilateral Π_i is naturally identified with the cyclic graph $\Box$, with 4 edges and 4 vertices connected successively. Denote by d_i the metric on $\Box$ induced from $(\Pi_i, \frac{d}{R(\gamma_i)})$.



Morse lemma: the curved quadrilateral (2)

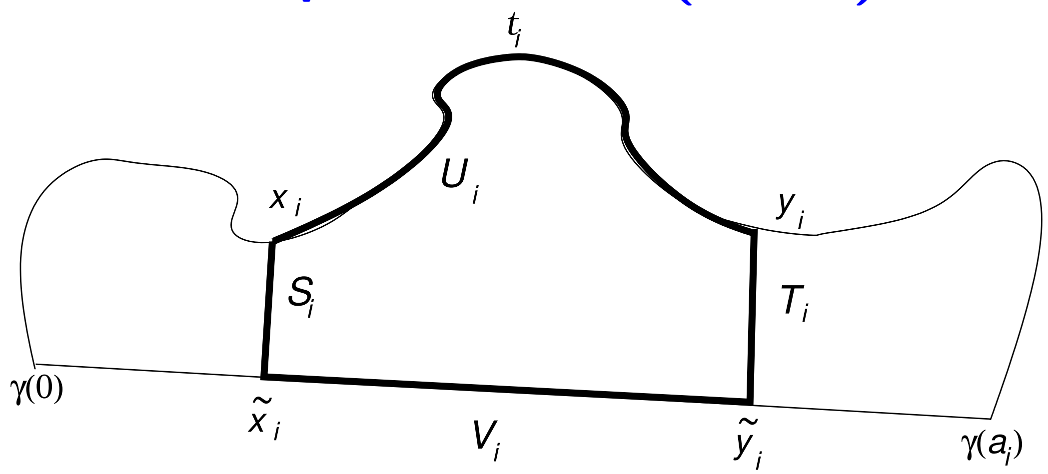


One of the sides of $(\square, d_i)$ is a C-quasigeodesic, and distance between its ends is $4C$, the adjacent sides have length $\leqslant 1$, and its opposite side is a geodesic, no longer than $4C + 2$. Therefore, **$\{d_i\}$ has a subsequence which converges uniformly to a pseudometric d on $\square$, and satisfies the 0-Gromov inequality.**

Let $U \subset (\square, d)$ be the limit of the curved side U_i, V the limit of the opposite side, and S, T the other two sides. Then $|U| = 4C$, $|V| \leqslant 4C + 2$, $|S|, |T| \leqslant 1$, and the metric space associated with $(\square, d)$ is a tree.

Let $t \in U$ be the limit of $t_i \in U_i$. Since $Cd(\gamma(x_i), t_i) \geqslant |x_i - t_i| = 2CR(\gamma_i)$, **the distance between t and the ends of U is $\geqslant 2$.**

Morse lemma: the curved quadrilateral (finale)



PROPOSITION: Let $\Psi : \square \longrightarrow \mathbb{T}$ be a map from a quadrilateral to a tree with U mapped to an interval of length $4C$, and adjacent edges to intervals of length $\leqslant 1$. Consider a point $t \in U$ such that the distance between t and the ends of U is > 1. **Then** $\Psi(t) \in \Psi(V)$. ■

We obtained $d(t, V) = 0$, but, by construction, $d_i(t_i, V_i) = 1$, which leads to contradiction. **Therefore,** $\lim_i \frac{R(\gamma_i)}{a_i} = 0$ **implies** $\lim_i R(\gamma_i) < R < \infty$**.**

Morse lemma: the end of the proof.

COROLLARY: Let $\gamma_i : [0, a_i \longrightarrow M$ be a sequence of C-quasigeodesics in a δ-hyperbolic space. Then $\lim R_i(\gamma_i) < \infty$.

Proof: $\lim_i \frac{R(\gamma_i)}{a_i} = 0$, as follows with the argument with the digon. Then $\lim R_i(\gamma_i) < \infty$, as follows from the case of triangle and quadrilateral. ∎

REMARK: Gromov and [BBI] give more constructible proofs, where the estimate on R is obtained as a function of C and δ.

REMARK: We have shown that a C-quasigeodesic belongs to an R-neighbourhood of a geodesic, connecting its ends: $S \subset T(R)$. **The same argument shows that $T \subset S(R)$.**