Metric spaces

lecture 20: Rips polyhedron

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Rips polyhedron

DEFINITION: A right-angle spherical simplex of diameter d is an intersection C_d of a sphere S^n of radius $2\pi^{-1}d$ in $\mathbb{R}^{n+1}$ and the coordinate sector $x_1 \geqslant 0, ..., x_{n+1} \geqslant 0$.

CLAIM 1: The distance from any vertex of C_d to any point of the opposite face is equal to d. ■

DEFINITION: Let M be a metric space. Its **Rips polyhedron** is a polyhedral space $P_d(M)$, obtained as follows. Its vertices are all $x \in M$. An n-simplex of $P_d(M)$ is spanned by any n points $m_1, ..., m_n$ in M such that $d(m_i, m_j) \leqslant d$. We put a geodesic metric on the Rips polyhedron in such a way that **every side has length d, and every simplex of $P_d(M)$ is isometric to a right-angle spherical simplex of diameter d.**

Rips polyhedron: the distance between vertices

THEOREM 1: Let M be a geodesic metric space, and $M \xrightarrow{\varphi} P_\delta(M)$ a natural embedding. **Then** $d(\varphi(x), \varphi(y)) = \delta\lceil\delta^{-1}d(x,y)\rceil$, where $\lceil a\rceil$ is the ceiling function, the smallest integer which is $\geqslant a$.

Proof. Step 1: Clearly, if $d(x,y) \leqslant \delta m$, we can connect x to y by a chain of geodesic segments of length $\leqslant \delta$: $[x, z_1], [z_1, z_2], ..., [x_{m-2}, y]$. In $P_\delta(M)$, these two points are connected by a sequence of 1-simplices, hence $d(\varphi(x), \varphi(y)) \leqslant \delta\lceil\delta^{-1}d(x,y)\rceil$. **It remains to show that $d(\varphi(x), \varphi(y)) \geqslant \delta\lceil\delta^{-1}d(x,y)\rceil$.**

Step 2: Let γ be a path in the Rips polyhedron starting and ending in vertices. Replacing each segment by the corresponding geodesic, we can always assume that any piece of γ which is contained in a simplex is a geodesic. Let A, B be neighbouring simplices, γ_1, γ_2 the corresponding segments of γ, with γ_1 starting in a vertex $v_a \in A$ and ending in b on its face, and γ_2 starting at $b \in B$ and ending in c. **We will replace $\gamma_1 \cup \gamma_2$ with another union of geodesic segments $\gamma_1' \cup \gamma_2'$, connected in b' and no longer than $\gamma_1 \cup \gamma_2$, also starting in v_a and ending in c, and such that b' is a vertex.** We use induction in dimension of A and B.

Rips polyhedron: the distance between vertices (2)

Step 3: If b belongs to a face of A which is adjacent to a, we replace A by this face, **ending up with b on the face which is opposite to a, or A being 1-dimensional.** In the second case, c belongs to a simplex which contains A as its face, and we replace γ_2 by a single segment connecting v_a and c, and γ_1 by a 0-geodesic connecting v_a to itself.

Step 4: Finally, suppose that b belongs to the opposite side D. If c is a vertex of B opposite to D, then $d(v_a, b') = d(c, b') = \delta$ for every point $b' \in D$, and we replace b by a vertex of D. If c is not a vertex, it lies on a face B_1 of B. We replace b by the endpoint b' of the geodesic perpendicular to D and starting in c. Since the faces B_1 and D are orthogonal, b' belongs to the intersection of these faces, which is a spherical right-angle simplex of codimension 1 in A and in B. Denote by A_1 the face of A which has $B_1 \cap D$ as its face. We replaced the segment $\gamma_1 \cup \gamma_2$ by a segment $\gamma_1' \cup \gamma_2'$ which belongs to a union $A_1 \cup B_1$ of two right-angle spherical simplices of smaller dimension. Using induction in $\dim A$, **replace $\gamma_1 \cup \gamma_2$ by a union of two geodesics which are no bigger in length and meet each other in a vertex of $P_d(M)$.**

Rips polyhedron: the distance between vertices (3)

Step 5: Steps 2-4 imply that for any piecewise geodesic path γ connecting vertices of $P_\delta(M)$, **there is another piecewise path γ_1 of length $\leqslant L(\gamma)$ obtained as a union of geodesics connecting vertices $x, z_1, ..., z_n, y$ of $P_\delta(M)$.** Then

$$d(\varphi(x), \varphi(y)) \geqslant L(\gamma_1) - \varepsilon = (n+2)\delta - \varepsilon,$$

where n is the minimal length of the chain $x, z_1, ..., z_n, y$ with all segments $\leqslant \delta$ and ε arbitrarily small. This implies that $d(\varphi(x), \varphi(y)) \geqslant \delta\lceil \delta^{-1} d(x,y)\rceil$. ■

$\delta, +$-quasiisometries

DEFINITION: A map $f: X \longrightarrow Y$ is called **a $\delta, +$-quasimetric map** if $d(f(x), f(y)) \leqslant d(x, y) + \delta$. **A $\delta, +$-quasiisometry** is a $\delta, +$-quasimetric map $f: X \longrightarrow Y$, such that a δ-neighbourhood of $f(X)$ contains Y.

LEMMA: A natural map $\varphi: M \longrightarrow P_\delta(M)$ of a geodesic space to its Rips polyhedron is a $\delta, +$-quasiisometry.

Proof: Indeed, $P_\delta(M)$ belongs to a δ-neighbourhood of $\pi(M)$, and $\operatorname{codiam} \varphi \leqslant \delta$. ■

PROPOSITION: $\delta, +$-quasiisometric spaces are quasi-isometric.

Proof. Step 1: Let N_x be a 2δ-separated δ-net in X, and $f: X \longrightarrow Y$ be δ-quasiisometry. **Then $f: N_X \longrightarrow f(N_X)$ is bi-Lipschitz** (Lecture 16).

Step 2: For any $x \in X$, we have $B_{f(x)}(2\delta) \supset f(B_x(\delta)) \cap f(X)$, because the codiameter of f is $\leqslant \delta$. This implies that $f(N_X)(2\delta) \supset f(N_X(\delta)) \cap f(X) = f(X)$. **Since $f(X)(\delta) = Y$, this implies thet $N_Y := f(N_X)$ is a 3δ-net.**

Step 3: We have constructed **bi-Lipschitz bijection between 3δ-nets in X, Y, which are therefore quasiisometric.** ■

Rips polyhedra of ε-nets

CLAIM: Let N_X be an ε-net in a geodesic space (X, d). Define the distance $d_{2\varepsilon}$ on X as $d_{2\varepsilon}(x, y) = \inf_S \sum d(x_i, x_{i+1})$, where the infimum is taken over all sequences $S = \{x_0 = x, x_1, ..., x_n = y\} \subset N_X$, such that $d(x_i, x_{i+1}) < 2\varepsilon$. **Then the metrics d_ε and d on N_X are bi-Lipschitz equivalent.**

Proof: Split the geodesic connecting x and $y \in N_x$ to $N = \lceil \varepsilon^{-1} d(x, y) \rceil$ intervals $[z_i, z_{i+1}]$ of length $\leqslant \varepsilon$. Let $x_i \in N_x$ be points which satisfy $d(x_i, z_i) < \varepsilon$. Then $d(x_i, x_{i+1}) \leqslant 3\varepsilon$, which gives $d(x, y) \leqslant d_{2\varepsilon}(x, y) \leqslant 3\varepsilon N \leqslant 4d(x, y)$. ■

Corollary 2: Let N_X be an ε-net in X, and $r \geqslant 2\varepsilon$. Then the natural embedding $P_r(N_X) \longrightarrow P_r(X)$ **is bi-Lipschitz to its image.** ■

Injective quasi-isometries of polyhedral spaces

LEMMA: Let A, B be simplicial polyhedral spaces with all simplices isometric to spherical right-angle simplices, and $\varphi: A \longrightarrow B$ an injective quasi-isometry, inducing homothety with the same coefficient on each polyhedron. **Then φ is bi-Lipschitz onto its image.**

Proof: Suppose that the simplices in A have diameter ε, in B diameter ε'. It suffices to prove that the restriction of φ to the set of vertices is bi-Lipschitz onto its image. On vertices, φ takes two points x, y with $d(x,y) = k\varepsilon$ to vertices satisfying $d(\varphi(x), \varphi(y) = l\varepsilon'$, where $\min(1, C^{-1}k) < l < Ck + v$, where C, v are fixed constants. Then $C^{-1}k \leqslant l \leqslant (v+C)k$ and $(v+C)^{-1}l \leqslant k \leqslant 2Cl$, hence φ is bi-Lipschitz. ■

Proposition 3: Let N_X, N_Y be ε-nets in geodesic spaces, and $f: N_X \longrightarrow N_Y$ a bi-Lipschitz map which satisfies $C^{-1}d(x,y) \leqslant d(f(x), f(y)) \leqslant Cd(x,y)$. Consider the natural map $\varphi: P_{2\varepsilon}(N_X) \longrightarrow P_{2C\varepsilon}(N_Y)$ inducing homothety with coefficient C on each spherical simplex. **Then φ is bi-Lipschitz to its image.**

Proof: This map is an injective quasi-isometry, because $P_{2\varepsilon}(N_X)$ is quasi-isometric to X, f is quasi-isometry, and $P_{2C\varepsilon}(N_Y)$ is quasi-isometric to Y. Then the previous lemma implies that φ is bi-Lipschitz to its image. ■

Rips polyhedra and decomposition of quasiisometries

DEFINITION: Two spaces X, Y are **+-quasiisometric** if there exists a $\delta, +$-quasiisometry $f: X \longrightarrow Y$ for some $\delta > 0$.

PROPOSITION: Let X, Y be quasiisometric geodesic spaces. Then **there exists a chain of maps connecting X to Y which are +-isometries, and quasiisometric embeddings $A \longrightarrow B$ of polyhedral spaces which are bi-Lipschitz to its image.**

Proof: Let N_X, N_Y be C-bi-Lipschitz ε-nets in X, Y. Then X is +-quasiisometric to $P_r(X)$, which is quasi-isometric to its sub-polyhedron $P_r(N_X)$ (Corollary 2). The natural map $\varphi: P_r(N_X) \longrightarrow P_{Cr}(N_Y)$ is bi-Lipschitz to its image by Proposition 3. The embedding $\operatorname{im}\varphi \longrightarrow P_{Cr}(Y)$ is quasi-isometry, because N_X is quasi-isometric to Y, and bi-Lipschitz by Proposition 3. ■

δ-hyperbolicity $+$-quasiisometries

PROPOSITION: Let $f: X \longrightarrow Y$ be an $\varepsilon, +$-quasiisometry. Assume that X is δ-hyperbolic **Then Y is $\delta + 8\varepsilon$-hyperbolic.**

Proof: Gromov inequality is equivalent to

$$\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta.$$

Since f is $\varepsilon, +$-quasimetric map, for any 4 points in $f(X)$, we have $\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta - 4\varepsilon$. Since f is $\varepsilon, +$-quasiisometry, we have $f(X)(\varepsilon) = Y$. Therefore, for any 4 points in Y, we have $\max(|bp| + |ac| - |cp| - |ab|, |ap| + |bc| - |cp| - |ab|) \geqslant -\delta - 8\varepsilon$. ■

δ-hyperbolicity and quasiisometries

THEOREM: Let X, Y be quasiisometric geodesic spaces, and X is Gromov hyperbolic. **Then Y is also Gromov hyperbolic.**

Proof: A quasiisometry can be decomposed onto a chain of of $+$-quasi-isometries emedding of polyhedral spaces which are bi-Lipschitz to its image. Therefore, the theorem is implied by the following

PROPOSITION: Let $X \xrightarrow{f} Y$ be a quasi-isometric embedding which is bi-Lipschits to its image. **Then X is Rips hyperbolic $\Leftrightarrow$ Y is Rips hyperbolic.**

Proof. Step 1: Let Y be Rips δ-hyperbolic, $T_1, T_2, T_3 \subset X$ the sides of a geodesic triangle connecting $a, b, c \in X$, and $S_1, S_2, S_3 \subset Y$ the sides of a geodesic triangle connecting $f(a), f(b), f(c)$. The distance between S_i and $f(T_i)$ is R-bounded by Morse lemma, and S_i belongs to a δ-neighbourhood of $S_j \cup S_k$ because Y is δ-hyperbolic. Then $f(T_j)$ belongs to a $\delta + R$-neighbourhood of $S_j \cup S_k$, and to a $\delta + 2R$-neighbourhood of $f(T_j) \cup f(T_k)$. Finally, since f is bi-Lipschitz, $f(T_j) \subset f(T_j)(\delta + 2R) \cup f(T_k)(\delta + 2R)$ implies $T_j \subset T_j(C\delta + 2CR) \cup T_k(C\delta + 2CR)$. **We have proven that X is Rips $C\delta + 2CR$-hyperbolic.**

δ-hyperbolicity and quasiisometries (2)

PROPOSITION: Let $X \xrightarrow{f} Y$ be a quasi-isometric embedding which is bi-Lipschits to its image. **Then X is Rips hyperbolic $\Leftrightarrow$ Y is Rips hyperbolic.**

Step 2: Conversely, assume that X is Rips δ-hyperbolic. Consider a minimising geodesic γ in Y, and split γ onto equal intervals $[y_i, y_{i+1}]$ of size between 4ε and 8ε. Since $Y = f(X)(\varepsilon)$, we can choose $z_i \in f(X)$ such that $d(z_i, y_i) \leqslant \varepsilon$. Then

$$(i-j)2\varepsilon \leqslant d(z_i, z_j) \leqslant (i-j)6\varepsilon.$$

Let $x_i := f^{-1}(z_i)$. Then

$$C^{-1}(i-j)2\varepsilon \leqslant d(x_i, x_j) \leqslant C(i-j)6\varepsilon. \quad (*)$$

Denote by T the union of geodesic segments connecting x_i together. Then (*) implies that T is a quasigeodesic.

Step 3: From Step 2 and Morse lemma, applied to X, we obtain that in an R-neighbourhood of any geodesic triangle $S_1 \cup S_2 \cup S_3$ in Y there exists an image of a geodesic triangle $T_1 \cup T_2 \cup T_3 \subset X$. **Since Y is δ-Rips hyperbolic, we have $f(T_i) \subset f(T_j)(C\delta) \cap f(T_k)(C\delta)$, which implies that**

$$S_i \subset S_j(C\delta + 2R) \cap S_j(C\delta + 2R)$$

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Hyperbolic groups

DEFINITION: Let G be a group, and $\{s_i\}$ a collection of generators. **The Cayley graph** of the pair $(G, \{s_i\})$ is the metric graph, with the set of vertices identified with G, and edges connecting g and gs_i. The length of all edges the Cayley graph is set to the same number t, usually $t = 1$.

DEFINITION: Let G be a group, S a collection of generators, and $\Gamma_{G,S}$ its Cayley graph. **The word metric** on G is defined as the restriction of the graph metric from $\Gamma_{G,S}$ to $G \subset \Gamma_{G,S}$.

DEFINITION: Let Γ be a group and S its generator set. We say that Γ is **Gromov hyperbolic** if its Cayley graph C_Γ is δ-hyperbolic, for some $\delta > 0$.

REMARK: Let d_S be the word metric on Γ. Since C_Γ is quasisometric to (Γ, d_S), **a group is Gromov hyperbolic if and only if the metric space (Γ, d_S) is Gromov hyperbolic.**

THEOREM: Let (Γ, S) be a hyperbolic group. **Then Γ is hyperbolic with any other set of generators.**

Proof: The group Γ with the word metric d_S is quasiisometric (in fact, bi-Lipschitz equivalent) to Γ with the word metric $d_{S'}$ associated with another set of generators. Therefore, their Cayley graphs are also quasiisometric. ■