Metric spaces

lecture 21: Hyperbolic groups are finitely presented

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February 27, 2022, 17:00

Hyperbolic groups (reminder)

DEFINITION: Let G be a group, and $\{s_i\}$ a collection of generators. **The Cayley graph** of the pair $(G, \{s_i\})$ is the metric graph, with the set of vertices identified with G, and edges connecting g and gs_i. The length of all edges the Cayley graph is set to the same number t, usually $t = 1$.

DEFINITION: Let G be a group, S a collection of generators, and $\Gamma_{G,S}$ its Cayley graph. **The word metric** on G is defined as the restriction of the graph metric from $\Gamma_{G,S}$ to $G \subset \Gamma_{G,S}$.

REMARK: Let d_S be the word metric on Γ. Since C_Γ is quasisometric to (Γ, d_S), **a group is Gromov hyperbolic if and only if the metric space (Γ, d_S) is Gromov hyperbolic.**

THEOREM: Let (Γ, S) be a hyperbolic group. **Then Γ is hyperbolic with any other set of generators.**

Finitely generated and finitely presented groups

DEFINITION: A group is called **finitely generated** if it is isomorphic a quotient $\frac{\mathbb{F}_n}{H}$ of a free group with finite number of generators. It is called **finitely presented** if H is the minimal normal subgroup of $\mathbb{F}_n$ containing a finite subset $R \subset \mathbb{F}_n$.

EXERCISE: Construct a finitely generated, not finitely presented group.

DEFINITION: Let Γ be a finitely generated group with generators $x_1, ..., x_n$. We say that **the word problem is solvable** if there exists an algorithm which takes two words W_1, W_2 in letters $x_1, ..., x_n$ and tells whether W_1 and W_2 represent the same element of Γ.

Finitely generated and finitely presented groups (2)

REMARK: In 1955, P. S. Novikov **constructed a group where the word problem is unsolvable.** Collins, Donald J. (1986), *"A simple presentation of a group with unsolvable word problem"*, Illinois Journal of Mathematics 30 (2): 230-234,
John Pedersen's "A Catalogue of Algebraic Systems" *An explicit example of a reasonable short presentation with insoluble word problem*

$$
\begin{array}{llll}
\langle & a,b,c,d,e,p,q,r,t,k & | & \\
& p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
& p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
& p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
& p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
& p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
& aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
& bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
& cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
& dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
& eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 & \rangle
\end{array}
$$

THEOREM: Any finitely generated hyperbolic group Γ is finitely presented. Moreover, the word problem in Γ is solvable.

Proof: Next lecture. ■

Finite polyhedral spaces

THEOREM: Let S be a contractible simplicial polyhedron and Γ a discrete group which freely and properly discontinuously acts on S by polyhedral isomorphisms. Assume that the quotient $X := S/\Gamma$ is compact. **Then $\Gamma = \pi_1(X)$ is finitely generated and finitely presented.**

Proof. Step 1: Let X_1 be the 1-skeleton of X, that is, the union of all 1-simplices. Clearly, any path in a simplicial space X is homotopic to a path along the edges (1-simplices). Therefore, **the natural map $\pi_1(X_1) \longrightarrow \pi_1(X)$ is surjective.** Since X_1 is finite, this implies that $\pi_1(X)$ is finitely-generated.

Step 2: Let W be a word in generators of $\pi_1(X)$, interpreted as a path in X_1. The relation $W = 1$ in $\pi_1(X)$ is interpreted as a map $\varphi: \Delta \longrightarrow X$ from a disk to X with its boundary equal to W. Any such disk is homotopy equivalent to a disk with the same boundary in the 2-skeleton X_2 of X, **which implies that $\pi_1(X_2) = \pi_1(X)$.** It remains to show that **the fundamental group of a finite cellular space is finitely presented.**

Finite polyhedral spaces (2)

Step 3: Since X_1 is a graph, it is homotopy equivalent to a bouquet of circles. Shrinking the corresponding edges, we can replace X_2 by a cellular space with only one vertex and the same fundametal group. From now on, **we assume that $X = X_2$ has only one vertex.**

Step 4: Let $D_1, ..., D_n$ be 2-cells of X_2, and W_i their boundaries. Then $W_i = 1$ are relations in $\pi_1(X_2)$. To prove that $W_1 = 1, ..., W_n = 1$ are the only relations in $\pi_1(X_2)$, we use Seifert-Van Kampen theorem. Assume that this statement is true in a space Y_k obtained by gluing $D_1, ..., D_k$ to X_1. We need to prove it for Y_{k+1}. We apply Seifert-Van Kampen to $A = Y_k$, $B = D_{k+1}$, $C = X_1 \supset A \cap B$. Seifert-Van Kampen gives $\pi_1(A \cup B) = \pi_1(A) \star_{\pi_1(C)} \pi_1(B)$. However, $\pi_1(A) \star_{\pi_1(C)} \pi_1(B)$ is $\pi_1(A)$ with an extra relation $D_{k+1} = 1$. ■

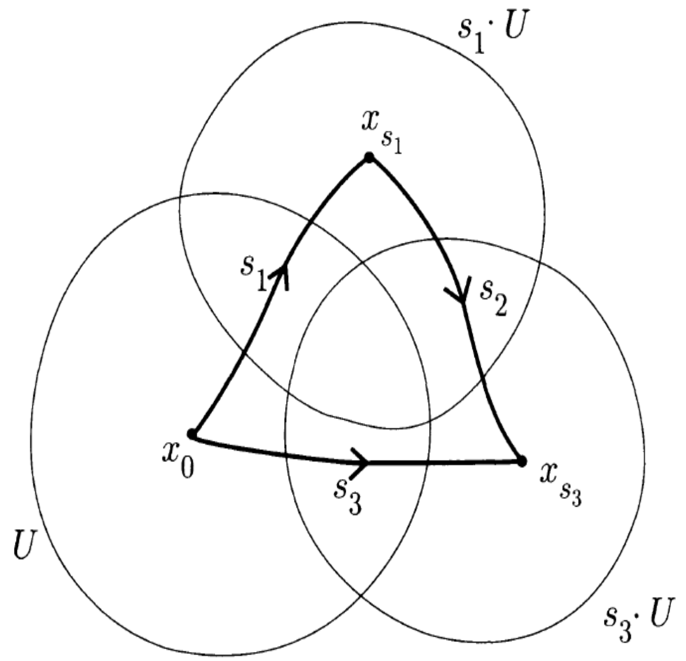
REMARK: We are going to prove the same result **when Γ is allowed to have finite stabilizers.**

Cocompact group action

THEOREM: Let M be a polyhedral metric space, and Γ a group acting on M properly discontinuously by isometries. Suppose that M/Γ is compact and simply connected. Let $U = B_x(R) \subset M$ be an open ball such that $\Gamma \cdot U = M$; suppose that it is contractible. Then

(i) **Γ is generated by a finite set $S = \{r \in \Gamma \ \ r(U) \cap U \neq \emptyset\}$.**

(ii) Let R be the set of all words $s_1 s_2 s_3^{-1}$ in $\mathbb{F}_S$ such that $s_1 s_3 s_3^{-1} = 1$ in Γ and $U \cap s_1(U) \cap s_3(U) \neq \emptyset$. **Then Γ is generated by S with the set of relations given by R.**



Cocompact group action (2)

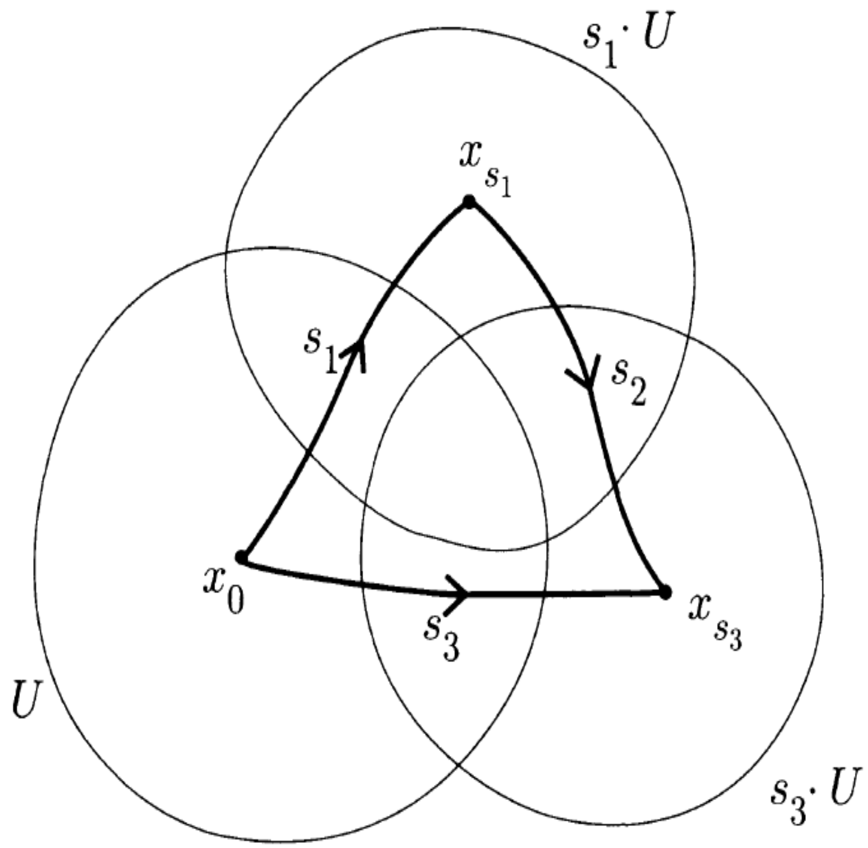
Proof. Step 1: Consider a path going from $x \in U$ to $\gamma(x)$. Then $\gamma \subset \bigcup_{i=1}^n \gamma_i(U)$, ordered monotonously along γ, with the adjacent $\gamma_i(U)$ satisfying $\gamma_i(U) \cap \gamma_{i+1}(U) \neq 0$, **which implies that $s_i \gamma_i \gamma_{i+1}^{-1} \in S$, and $\gamma = \prod_{i=1}^n s_i$.**

Step 2: Consider a 2-dimensional simplicial complex C obtained from the Cayley graph of (Γ, S) by attaching a disk Δ_r with boundary glued to any $r \in R$. From the same argument using Seifert-Van Kampen theorem, **to show that Γ is given by relations in S it would suffice to prove that $\pi_1(C) = 1$.**

Step 3: Fix $x \in M$. We embed the Cayley graph of Γ to M taking $\gamma \in \Gamma$ to $\gamma(x)$ and connecting these points by geodesics. Since $\pi_1(M) = 0$, all 2-cells in C are contractible in M, hence we can glue them in, **extending the embedding of the Cayley graph to a Γ-invariant map $\Phi: C \longrightarrow M$.**

Step 4: It would suffice to show that any path σ in 1-skeleton of C is contractible in C. Since M is simply connected, **the restriction of Φ to σ can be extended to a map $\varphi: \Delta \longrightarrow M$,** where Δ is a disc, and φ restricted to its boundary is $\Phi|_\sigma$.

Cocompact group action (3)



Step 5: Choose a triangulation T of Δ such that every vertex of T is mapped to $\gamma(x)$, for some $\gamma \in \Gamma$; this is possible to do using the contraction of U to x. Every triangle of this triangulation corresponds to s_1, s_2, s_3 such that $s_1 s_3 s_3^{-1} = 1$, and $U \cap s_1(U) \cap s_3(U) \neq \emptyset$, hence **one can contract σ to a point sequentially using contractions which come from $s_1 s_3 s_3^{-1} = 1$.** ■

REMARK 1: The same proof will work without compactness of M/Γ, if we assume that $\Gamma \cdot U = M$ and that the set of $\gamma \in \Gamma$ such that $U \cap \gamma(U) \neq \emptyset = infty$.

Rips polyhedron is weakly contractible

THEOREM: (Rips-Gromov) Let X be a Gromov δ-hyperbolic geodesic space. **Then its Rips polyhedron $P_r(X)$ satisfies $\pi_i(P_r(X)) = 0$, for any $i > 0$ and any $r \geqslant 4\delta$.**

Proof. Step 1: Any map $S^n \longrightarrow P_r(X)$ is homotopic to a map to a finite sub-polyhedron. Therefore, **it suffices to show that any finite polyhedral subset $D \subset P_r(X)$ can be contracted to a point in $P_r(X)$.**

Step 2: Any finite polyhedron $D \subset P_r(X)$ with vertices in a subset $U \subset X$ of diameter less than d is contractible, because $P_r(X)$ **contains a simplex spanned by any subset $P_r(U)$.**

Step 3: Let $p \in X$ be a point, $D \subset P_r(X)$ finite polyhedon, and y_0 a vertex of D such that $d(p, y_0) \geqslant d(p, z)$ for any other vertex $z \in D$. Take a point y_1 on the minimizing geodesic $[p, y_0]$, such that $d(y_1, y_0) \geqslant \frac{1}{2}r$. **It would suffice to prove that D can be homotopied to the polyhedron D_1 with all vertices of D are the same, except y_0, and y_0 is replaced by y_1.**

Step 4: The polyhedron D_1 exists if for each $z \in X$, one has $d(y_1, z) \leqslant \min(r, d(y_0, z))$. The homotopy between D to D_1 is obtained by constructing a polyhedron D_2 which has D and D_1 as components of its boundary.

Rips polyhedron is weakly contractible (2)

Step 5: For each face of D and D_1 which does not contain y_0, y_1, this face is a face of D_2, and for each k-dimensional simplex $(y_0, z_1, ..., z_k)$ of D_0 which we add the $k+1$-dimensional simplex $(y_0, y_1, z_1, ..., z_k)$ to D_2. Then D can be homotopied to D_1 in D_2. Therefore, Rips-Gromov theorem is implied by the following lemma.

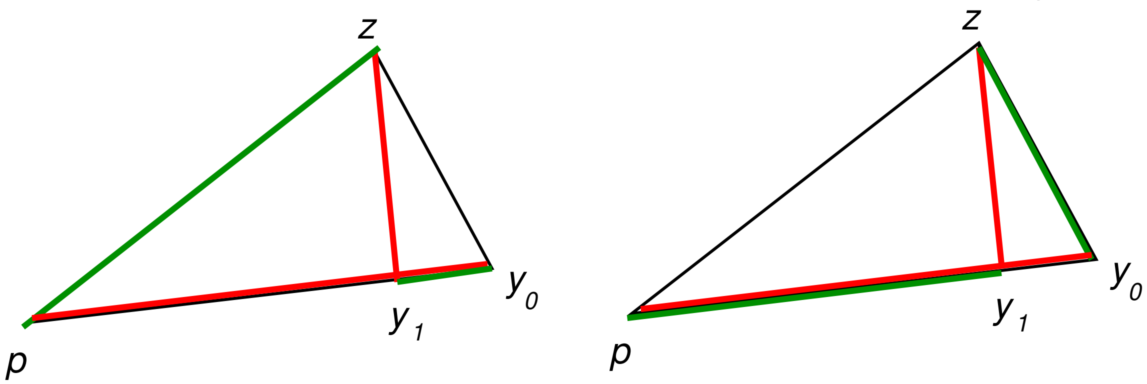
LEMMA 1: Let p, y_0, z, $d(p, y_0) \geqslant d(p, z)$ be points of a geodesic space which is δ-hyperbolic, $r \geqslant 2\delta$, and y_1 a point on a minimizing geodesic $[p, y_0]$ which satisfies $d(y_1, y_0) \geqslant \frac{1}{2}r$. **Then** $d(y_1, z) \leqslant \min(d, d(y_0, z))$**.**

Proof: Next slide

Rips polyhedron is weakly contractible (3)

LEMMA 1: Let p, y_0, z, $d(p, y_0) \geqslant d(p, z)$ be points of a geodesic space which is δ-hyperbolic, $r \geqslant 2\delta$, and y_1 a point on a minimizing geodesic $[p, y_0]$ which satisfies $d(y_1, y_0) \geqslant \frac{1}{2}r$. **Then** $d(y_1, z) \leqslant \min(d, d(y_0, z))$**.**

Proof: Gromov inequality applied to the quadruple p, y_0, z, y_1.



implies that $\delta - |zy_1| - |py_0| + |y_1 y_0| + |pz| \geqslant 0$ or $\delta - |zy_1| - |py_0| + |zy_0| + |py_1| \geqslant 0$. This gives $\delta + |zp| - |py_1| - |zy_1| \geqslant 0$ or $\delta + |zy_0| - |zy_1| - |y_0 y_1| \geqslant 0$.

In the first case, using $|py_0| \geqslant |py_1| + \frac{1}{2}r \geqslant |pz|$, we obtain

$$|zy_1| \leqslant \delta + |zp| - |py_1| \leqslant \delta + |py_1| + \frac{1}{2}r - |py_1| = \delta + \frac{1}{2}r \leqslant r.$$

In the second case,

$$|zy_0| \geqslant |zy_1| + |y_0 y1| - \delta \geqslant |zy_1| + \frac{r}{2} - \delta \geqslant |zy_1|. \blacksquare$$

Hyperbolic groups are finitely generated

Applying Remark 1 to the action of Γ on the Rips polyhedron of the Cayley graph of Γ, we obtain

COROLLARY: Let Γ be a Gromov hyperbolic group. **Then Γ is finitely presented.**

Proof: For any $r \geqslant 2\lceil\delta\rceil$, the Rips polyhedron $P_r(X)$ of Γ is simply connected, and the action of Γ on $P_r(X)$ satisfies assumptions of Remark 1 above. ■

REMARK: There exists a finitely generated subgroup of a hyperbolic group **which is not finitely presented:** *N. Brady, Branched coverings of cubical complexes and subgroups of hyperbolic groups, 60 (2), 461-480, 1999.*

Properties of hyperbolic groups

EXERCISE: Let Γ be a group acting properly discontinuously on a contractible polyhedral space M, such that M/Γ is compact. **Prove that $H^i(M/\Gamma, \mathbb{Q}) = H^i(\Gamma, \mathbb{Q})$.**

COROLLARY: Let Γ be a hyperbolic group. **Then $\dim H^i(\Gamma, \mathbb{Q}) < \infty$.** ■

THEOREM: Let Γ be a hyperbolic group. **Then the number of conjugacy classes of elements of finite order in Γ is finite.**

Proof: By Brower theorem, every element γ fixes a point $z \in P_r(\Gamma)$, and conjugate elements correspond to the same points in $P_r(\Gamma)/\Gamma$. This defines a bijective correspondence between the conjugacy classes of elements of finite order, and the strata $P_r(\Gamma)/\Gamma$ associated with the stratification of $P_r(\Gamma)$ by stabilizers of the action of Γ. Since **these strata are polyhedral strata in the baricentric decomposition of the polyhedral space $P_d(\Gamma)/\Gamma$, there are only finitely many.** ■