Metric spaces

lecture 22: Word problem in hyperbolic groups

Misha Verbitsky

IMPA, sala 236

March 3, 2022, 17:00

Groups with relators of length at most 3

DEFINITION: Let Γ be a group presented by the set $S = \{s_1, s_1^{-1}, ..., s_n, s_n^{-1}\}$, with relation set $R = \{W_1, W_2, ..., W_n\}$ given by reduced words in elements of S. The words W_i are called **the relators** of Γ. In this case we write $\Gamma = \langle S|R\rangle$.

REMARK: $\Gamma = \mathbb{F}_n/\Gamma_R$, where $\Gamma_{\mathbb{R}} \subset \mathbb{F}_n$ is a subgroup generated by xW_ix^{-1} for all $x \in \mathbb{F}_n$.

DEFINITION: We are interested in **groups with relators of length 3,** that is, with all relators having form $W_i = u_iu_ju_k$, where $u_i, u_j, u_k \in S$. In this situation, **we will always add $W_i^{-1} = u_k^{-1}u_j^{-1}u_i^{-1}$ to the set of relators.**

REMARK: Loops in the Cayley graph of Γ are relations of form $x_{i_1}x_{i_2}...x_{i_N} = 1$. **If every relation in Γ has form xW_ix^{-1}, with $|W_i| = 3$, this means that every loop can be cut onto triangles with side 1,1,1.**

Area of a contractible loop

DEFINITION: Let $\Gamma = \langle S|R\rangle$ be a group with relators of length at most 3. **Area** of a loop in the Cayley graph of Γ is the minimal number of triangles, obtained if we cut this loop on triangles.

REMARK: A loop of length n starting in x in the Cayley graph is a relation of form $xt_1x^{-1}xt_2x^{-1}...xt_rx^{-1}$, where $t_i \in S$. Then **the area of this loop is the smallest number of length 3 relators $W_1, ..., W_k$ such that** $xt_1x^{-1}xt_2x^{-1}...xt_rx^{-1} = a_1W_1a_1^{-1}...a_kW_ka_k^{-1}$ in the free group generated by S.

THEOREM: Let $\Gamma = \langle S|R\rangle$ be a group with relators of length at most 3, and Ψ a computable function such that every loop of length n has area $\leqslant \Psi(n)$. **Then the word problem is solvable in Γ.**

Word solvable groups

THEOREM: Let $\Gamma = \langle S|R\rangle$ be a group with relators of length at most 3, and Ψ a computable function such that every loop of length n has area $\leqslant \Psi(n)$. **Then the word problem is solvable in Γ.**

Proof. Step 1: Let the loop $\gamma := x_{i_1}x_{i_2}...x_{i_n}$, be cut onto triangles $w_{a_1b_1c_1}$, ..., with each triange $w_{a_ib_ic_i}$ having sides $a_i, b_i, c_i \in S$ starting at the vertex $\psi(i)$ in the Cayley graph; in other words, $x_{i_1}x_{i_2}...x_{i_n} = \prod_i w_{a_ib_ic_i}^{\psi(i)}$. Denote $\psi_i\psi_{i-1}^{-1}$ by h_i; this is the element connecting a triangle to the next one. Either the loop γ goes ahead from one triangle and back by the track in opposite direction, or the triangles are adjacent and h_i has length $\leqslant 2$. Let now $L_k := \prod_{i=1}^k h_k$ be a path connecting 1 and ψ_k. Using induction, we denote by g_k a loop $g_{k-1}L_kw_{a_kb_kc_k}L_k^{-1}$. This is a loop in the Cayley graph of Γ obtained by gluing the triangles from the 1-st up to k-th.

Step 2: The loop $x_{i_1}x_{i_2}...x_{i_N}$ can be obtained by going aroung the triangles $w_{a_ib_ic_i}^{\psi(i)}$ successively. **This gives relation $x_{i_1}x_{i_2}...x_{i_n} = g_N$ in $\mathbb{F}_S$.**

Word solvable groups (2)

Step 3: To prove that the word problem is solvable, we need to be able to write an algorithm able to represent any word $x_{i_1}x_{i_2}...x_{i_n}$ which is equal to 1 in Γ as a product of relators, $x_{i_1}x_{i_2}...x_{i_n} = \prod_i w_{a_ib_ic_i}^{\psi(i)}$. Consider all sequences of form $g_1 = L_1 w_{a_1b_1c_1} L_1^{-1}$, $g_2 = g_1 L_2 w_{a_2b_2c_2} L_2^{-1}$, ..., $g_N = g_{N-1} L_N w_{a_Nb_Nc_N} L_N^{-1}$, where $N \leqslant \Psi(n)$. There are finitely many such sequences, because on each step we take finitely many choices: choose a triangle $w_{a_ib_ic_i}$ and the word $h_i = \psi_i \psi_{i-1}^{-1}$ of length $\leqslant 2$. Therefore, to recover all possible relations $x_{i_1}x_{i_2}...x_{i_n} = \prod_i w_{a_ib_ic_i}^{\psi(i)}$, we need to take only $(2d + d')^{\Psi(N)}$ comparisons, where $d = |S|$ and $d' = |W|$. ■

Word problem in hyperbolic groups

THEOREM: Let Γ be a Gromov hyperbolic group. **Then the word problem in Γ is solvable.**

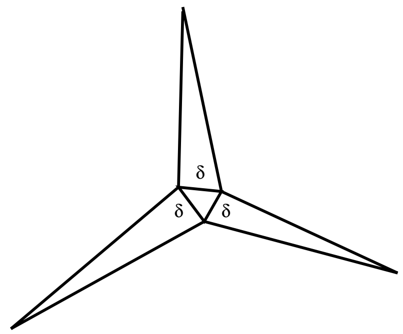
Proof. Step 1: Choose a presentation $\Gamma = \langle S|R\rangle$ with all relators of length 3. Suppose that the Cayley graph of Γ is Rips δ-hyperbolic. **It would suffice to show that the area of a path with perimeter P obtaned from N geodesic segments is bounded by $const NP + const' N$.** In this case the word problem is solvable (Porposition 1). The inequality bounding the area of a loop by a polynomial on its perimeter is called **the isomerimetric inequality.**

Step 2: Let C be the maximal area of a geodesic triangle with all sides $\leqslant \delta$. The number of such triangles is finite, hence C is finite.

Step 3: Let $\triangle$ be a geodesic triangle in the Cayley graph of Γ with perimeter P. We say that $\triangle$ is **δ-degenerate** if it belongs to a δ-neighbourhood of one of its sides. **the area of a degenerate triangle is bounded by $C\lceil\delta^{-1}P\rceil$.** Indeed, $\triangle$ can be cut onto $\lceil\delta^{-1}P\rceil$ triangles with side $\leqslant \delta$.

Word problem in hyperbolic groups: the isoperimetric inequality

Step 4: δ-hyperbolicity implies that any geodesic triangle can be cut onto 2 δ-degenerate triangles and one triangle with 3 sides $\leqslant \delta$.



Therefore, **the area of a triangle is $\leqslant C\lceil \delta^{-1}P\rceil + C \leqslant C\delta^{-1}P + 2C$, where P is its perimeter.**

Step 5: Let γ be a path of length P in the Cayley graph obtained from N geodesic segments, and γ' the path with $\lceil \frac{N}{2}\rceil$ geodesic segments, obtained from γ by replacing every successive union of odd and even segments by a geodesic. Then

$$\operatorname{Area}(\gamma) - \operatorname{Area}(\gamma') \leqslant C\lceil \delta^{-1}[\operatorname{Per}(\gamma) - \operatorname{Per}(\gamma')]\rceil + 2C\lceil \delta^{-1}\operatorname{Per}(\gamma')\rceil \leqslant \\ \leqslant 2C\left\lceil \frac{N}{2}\right\rceil + C\delta^{-1}\operatorname{Per}(\gamma)$$

which implies $\operatorname{Area}(\gamma) \leqslant 2C\delta^{-1}N\operatorname{Per}(\gamma) + 2CN$. ■