Home assignment 1: Lie groups

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor.

Definition 1.1. A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G acts on a manifold M if the group action is given by the smooth map $G \times M \longrightarrow M$.

Exercise 1.1. Prove that the Lie group $SL(n, \mathbb{R})$ is connected.

Exercise 1.2. Prove that the Lie group $O(1, 2)$ has 4 connected components.

Definition 1.2. Let $W \subset \operatorname{End}(V)$ be a vector space. Suppose that the exponent map defines a diffeomorphism between a neighbourhood of 0 in W and a neighbourhood of unity in a closed subgroup $G \subset GL(V)$. Then G is called a **matrix Lie group**.

Exercise 1.3. Let $G \subset GL(V)$ be a matrix Lie group, locally identified with exponents of $w \in W \subset \operatorname{End}(V)$. Prove that $W = T_e G$.

Exercise 1.4. Prove that the unitary group $U(n) \subset GL(\mathbb{C}^n)$ is a matrix Lie group.

Exercise 1.5. Let $G \subset GL(V)$ be a matrix Lie group, generated by exponents of $w \in W \subset \operatorname{End}(V)$, where $W = T_e G$. Let $Z \subset G$ be the kernel of the adjoint action of G on W.

a. Prove that Z is central (commutes with all G).

b. Prove that all central elements belong to Z.

Exercise 1.6. Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic vector space (a space equipped with a non-degenerate bilinear antisymmetric form ω), and $Sp(2n)$ the group of all endomorphisms of $\mathbb{R}^{2n}$ preserving the symplectic form. Prove that $Sp(2n)$ acts transitively on the space $\mathbb{R}^{2n} \backslash 0$.

Exercise 1.7. Prove that the special unitary group $SU(n)$ acts transitively on the projective space $\mathbb{C}P^{n-1}$. Find the stabilizer of a point.