Home assignment 2: Fundamental group and orientability

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor.

Definition 2.1. Determinant bundle on an n-dimensional manifold is the bundle $\Lambda^n(M)$ of antisymmetric n-linear forms. A manifold is called **orientable** if its determinant bundle is trivial.

Exercise 2.1. Let M be an n-manifold. Prove that the bundle $\Lambda^n(M)$ has rank 1.

Exercise 2.2. Let M be an almost complex manifold. Prove that it is orientable.

Exercise 2.3. Let $S \subset \mathbb{R}^3$ be a smooth compact 2-dimensional submanifold of $\mathbb{R}^3$. Prove that it is orientable.

Exercise 2.4. Let M be a manifold with fundamental group isomorphic to $\mathbb{Q}$. Prove that M is orientable.

Exercise 2.5. Let G be a Lie group. Prove that its fundamental group is commutative.

Definition 2.2. A covering is a continuous map $\pi : M \longrightarrow M_1$ such that for each $x \in M_1$ there exists a neighbourhood $U \subset M_1$ such that $\pi : \pi^{-1}(U) = U \times S$, and the map $\pi : \pi^{-1}(U) = U \times S \longrightarrow U$ is a projection.

Exercise 2.6. Find an example of a map $\pi : M \longrightarrow M_1$ which is locally a diffeomorphism, but not a covering.

Exercise 2.7. Prove that any non-oriented manifold M_1 has a 2-sheeted covering $\pi : M \longrightarrow M_1$ such that M is oriented.

Exercise 2.8. Let $\pi : M \longrightarrow M_1$ be a local diffeomorphism, such that every point has exactly k preimages. Prove that π is a covering or find a counterexample.

Exercise 2.9. Let M be an oriented compact 2-dimensional which admits a pseudo-Riemannian metric of signature (1,1). Prove that M has a 4-sheeted covering with trivial tangent bundle.

Exercise 2.10. Prove that an non-oriented compact 2-dimensional does not admit a pseudo-Riemannian metric of signature (1,1), or find a counterexample.