Home assignment 3: Convergence for Lipschitz maps.

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor.

Remark 3.1. For all metric spaces in this assignment, we assume that they admit a countable dense set (that is, are "second countable").

Definition 3.1. **A path** in a metric space M is a continuous map $\gamma:\ [0,1]\longrightarrow M$ **Length** of a path is $\sup_{0=x_1<x_2<...<x_n=1\in[0,1]}\sum_{i=1}^{n-1} d(\gamma(x_i),\gamma(x_{i+1}))$.

Exercise 3.1. Let $\gamma:\ [0,1]\longrightarrow\mathbb{R}^2$ be a path of length l, with $d(\gamma(0),\gamma(1))>l-\varepsilon$. Prove that the image of γ is contained in a rectangle of size less than $\varepsilon\times(l+\varepsilon)$.

Exercise 3.2. Let $\gamma:\ [0,1]\longrightarrow\mathbb{R}^2$ be a path of finite length. Prove that the image of γ has measure 0.

Hint. Use the previous exercise.

Definition 3.2. Let M be a metric space, and $\operatorname{Map}(X,M)$ the set of all maps from a set X to M. Define the metric on $\operatorname{Map}(X,M)$ using $d(f_1,f_2)=\sup_{x\in X} d(f_1(x),f_2(x))$. The corresponding topology is called **the uniform topology** on $\operatorname{Map}(X,M)$.

Exercise 3.3. Prove that the length of a path is continuous in the uniform topology on the space $\operatorname{Map}_C([0,1],M)$ of C-Lipschitz paths.

Exercise 3.4. Let M be a manifold with a marked point m, and $\operatorname{Map}((S^1,0),(M,m))$ the space of maps putting $0\in S^1$ to $m\in M$. We equip $\operatorname{Map}((S^1,0),(M,m))$ with a uniform topology. Denote by W the set of connected components of $\operatorname{Map}((S^1,0),(M,m))$. Consstruct a bijective equivalence between W and the fundamental group $\pi_1(M)$.

Definition 3.3. Let M be a topological space, and $U\subset M$ an open subset. Given $x\in X$, we define a subset $S_{x,U}\subset\operatorname{Map}(X,M)$ as the set of all maps putting x to U. **Tychonoff topology**, also called **topology of pointwise convergence**, is topology where open subsets are obtained by finite intersections and all unions of $S_{x,U}$ for all possible x and U.

Exercise 3.5. Let $\{f_i\}\in\operatorname{Map}(X,M)$ be a sequence of maps, with M being a topological space. Prove that $\{f_i\}$ converges to f in Tychonoff topology if and only if for each $x\in X$ one has $\lim_i f_i(x)=f(x)$.

Remark 3.2. Tychonoff theorem claims that $\operatorname{Map}(X,M)$ with Tychonoff topology is compact whenever M is compact. This is an important and difficult statement. Please make yourself aware of its proof (Wikipedia can help).

Exercise 3.6. Let X,M be metric spaces, $C>0$ a real number, and $\operatorname{Map}_C(X,M)$ the space of C-Lipschitz maps. Assume that X is compact. Prove that the uniform topology on $\operatorname{Map}_C(X,M)$ is equivalent to the Tychonoff topology.

Exercise 3.7. Prove that $\operatorname{Map}_C(X,M)$ is closed in $\operatorname{Map}(X,M)$ with Tychonoff topology.

Exercise 3.8. Let X,M be compact metric spaces, and $C>0$ a real number. Using Tychonoff theorem, prove that $\operatorname{Map}_C(X,M)\subset\operatorname{Map}(X,M)$ is compact in uniform topology.