Home assignment 4: tensor product.

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor.

Definition 4.1. Let S be a set. Define **vector space, freely generated by** S, as the space of functions $\psi: S \longrightarrow k$ which are equal zero outside of a finite subset in S.

Remark 4.1. This is the same as a space with basis enumerated by elements of S.

Definition 4.2. Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v_1' + v_2') - v \otimes v_1' - v \otimes v_2'$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

Exercise 4.1. Prove that $\dim(V \otimes W) = \dim V \cdot \dim W$.

Definition 4.3. Rank of a tensor $\psi \in V^{\otimes n}$ is the smallest number of tensor monomials $v_1 \otimes v_2 \otimes ... \otimes v_n$ such that ψ can be represented by a sum of these monomials.

Exercise 4.2. Let V be a 10-dimensonal vector space over $\mathbb{R}$. Find a tensor of rank 3 in $V^{\otimes 2}$ or prove it does not exist.

Exercise 4.3. Let V, W be vector spaces, and $\Psi_{V,W}: V^* \otimes W \longrightarrow \operatorname{Hom}(V, W)$ maps $\lambda \otimes w \in V^* \otimes W$ to an endomorphism which takes v to $\lambda(v)w$.

a. Prove that $\Psi_{V,W}$ is always injective.

b. Prove that it is an isomorphism when V, W are finite-dimensional.

Exercise 4.4. Prove that $\Psi_{V,W}: V^* \otimes W \longrightarrow \operatorname{Hom}(V, W)$ is an isomorphism, or find a counterexample, when

a. $\dim V$ is finite, $\dim W$ is infinite.

b. $\dim V$ is infinite, $\dim W$ is finite.

Exercise 4.5. Let V be a vector space. Consider a natural map Φ_V from $\underbrace{V^* \otimes V^* \otimes ... \otimes V^*}_{n \text{ times}}$ to the space of n-linear forms on V, taking $\lambda_1 \otimes \lambda_2 \otimes ... \otimes \lambda_n \in \underbrace{V^* \otimes V^* \otimes ... \otimes V^*}_{n \text{ times}}$ to a form $v_1, v_2, ..., v_n \longrightarrow \lambda_1(v_1)\lambda_2(v_2)...\lambda_n(v_n)$.

a. Prove that Φ_V is an isomorphism when V is finite-dimensional.

b. Prove that Φ_V is always injective

c. Prove that Φ_V is always an isomorphism, or find a counterexample.

Exercise 4.6. Let Φ be an n-linear form on a space V, and $A \in \operatorname{End}(V)$, with $B_t := e^{tA}$. Prove that Φ is B_t-invariant for all $t \in \mathbb{R}$ if and only if for any $v_1, ..., v_n \in V$, one has $\Phi(A(v_1), v_2, v_3, ..., v_n) + \Phi(v_1, A(v_2), v_3, ..., v_n) + \Phi(v_1, v_2, A(v_3), ..., v_n) + ... + \Phi(v_1, v_2, v_3, ..., A(v_n)) = 0$.