Home assignment 5: isometries of $\mathbb{H}^2$.

Rules: This is a class assignment for the next week. Please solve all exercises and discuss your solutions with your monitor.

Exercise 5.1. Let $\Lambda = \mathbb{Z}^3 \subset \mathbb{R}^3$ be the integer lattice, q an integer-valued quadratic form on Λ of signature $(1,2)$. Let $SO^+(\Lambda)$ be the subgroup of $SO^+(1,2)$ preserving Λ. Prove that $SO^+(\Lambda) \subset SO^+(1,2)$ is a discrete subgroup.

Definition 5.1. Order of $A \in GL(n)$ is the smallest positive integer such that $A^k = \mathsf{Id}$.

Exercise 5.2. Let A be an element of finite order k in $GL(2,\mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Exercise 5.3. Let $A \subset SL(3,\mathbb{Z})$ be an element of finite order k in $SL(3,\mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Hint. Use the previous exercise.

Remark 5.1. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $V^+ := \{v \in V \ \mid\ q(v) > 0\}$, and $\mathbb{P}V^+$ its projectivisation. Then $\mathbb{P}V^+ = SO^+(1,2)/SO(1)$, giving $\mathbb{P}V^+ = \mathbb{H}^2$; this is one of the standard models of a hyperbolic plane.

Definition 5.2. Let $l \subset V$ be a line, that is, a 1-dimensional subspace. The property $q(x,x) < 0$ for a non-zero $x \in l$ is written as $q(l,l) < 0$. A line l with $q(l,l) < 0$ is called **negative line**, a line with $q(l,l) > 0$ is called **positive line**.

Remark 5.2. Negative lines correspond to geodesics in $\mathbb{P}V^+ = \mathbb{H}^2$ (Lecture 8).

Exercise 5.4. Let γ_1, γ_2 be geodesics on a hyperbolic plane, and l_1, l_2 the corresponding negative lines.

a. Prove that l_1 is orthogonal to l_2 if and only if γ_1 is orthogonal to γ_2.

b. Prove γ_1 intersects γ_2 if and only if the 2-plane $\langle l_1, l_2\rangle$ generated by l_1, l_2 has signature (0,2).

c. Prove that γ_1 and γ_2 passes through the same point on the absolute if and only if the 2-plane generated by l_1, l_2 has degenerate scalar product.

d. Prove that the angle between γ_1 and γ_2 divides $\frac{2\pi}{k}$ if and only if the angle between l_1, l_2 in $\langle l_1, l_2\rangle$ divides $\frac{2\pi}{k}$.

e. Prove that the angle between γ_1 and γ_2 is equal to the angle between l_1, l_2 in $\langle l_1, l_2\rangle$.

Exercise 5.5. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integer coefficients, $h \in SO^+(1,2)$ a hyperbolic element with integer coefficients, and $P_h(t)$ its characteristic polynomial. Prove that $P_h(t)$ has precisely 1 rational root.