Complex manifolds in dimension 1: end-term exam

Rules: Every student receives from me a list of 8 exercises (chosen randomly), and has to solve as many of them as you can by due date (February 26, 2020). Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). You should come to IMPA February 26, 2020, 14:00 and find me (room 232 or 344).

The final score N is obtained by summing up the points from the exam problems and the class tests, using the formula $N = 10e + t/2$, where t is sum of class tests, e the points for exam problems. Marks: C when $20 \leqslant N < 30$, B when $30 \leqslant N < 40$, A when $40 \leqslant N < 50$, A+ when $N > 50$.

1 Almost complex structures and holomorphic functions

Exercise 1.1. Let f be a smooth real function on a disk D such that $dId(f)$ is a nowhere degenerate 2-form of positive orientation. Prove that f cannot have a maximum anywhere on D.

Definition 1.1. Holomorphic differential on an almost complex manifold is a closed $(1,0)$-form.

Exercise 1.2. Let M be a simply connected almost complex manifold, and η a holomorphic differential with no zeros. Prove that M admits a holomorphic submersion to $\mathbb{C}$.

Exercise 1.3. Let M be a simply connected almost complex manifold, and η a non-zero holomorphic differential. Prove that M admits a non-constant holomorphic map to $\mathbb{C}^* = \mathbb{C}\backslash 0$.

Exercise 1.4. Let M be an almost complex manifold, and $\phi : M \longrightarrow \mathbb{R}$ a function which satisfies $dId(\phi) = 0$. Prove that M admits a non-zero holomorphic differential.

Exercise 1.5 (2 points). Let $\Gamma \subset \operatorname{Aut}(\Delta)$ be a discrete, cyclic subgroup in the group of automorphisms of a disk. Prove that Δ admits a Γ-invariant holomorphic differential or find a counterexample.

Exercise 1.6 (2 points). Let M_0 be a Riemann surface equipped with a holomorphic action of a group $\Gamma = \mathbb{Z}$, generated by an automorphism γ. Consider a nowhere vanishing function ϕ on M_0 such that $\gamma^*(\phi) = \mathsf{const}\cdot\phi$ and $dId\phi = 0$. Prove that M admits a Γ-invariant holomorphic differential.

Exercise 1.7. Let M be a Riemannian 2-manifold, and $f : M \longrightarrow M$ a conformal map preserving the Riemannian volume. Prove that f is an isometry.

Exercise 1.8 (2 points). Let f be a real-valued smooth function on a complex manifold which satisfies $dId(f) = 0$. Prove that f is a real part of a holomorphic function or find a counterexample.

Exercise 1.9. Let f be a holomorphic function on a disk Δ. Prove that $\int_\Delta f\omega = \pi f(0)$, where $\omega = dx \wedge dy$ is the standard volume form.

Exercise 1.10. Let M be a simply connected complex manifold, and θ a non-zero exact 1-form such that $d(I\theta) = 0$. Prove that M admits a non-constant holomorphic function.

Exercise 1.11. Let f be a holomorphic function on a Riemann surface such that $|f|$ and df is nowhere zero. Prove that $|f|$ is smooth. Prove that $dId(|f|)$ is a nowhere degenerate 2-form.

Exercise 1.12. Let f be a non-constant holomorphic function on a disk, continuous on the boundary. Prove that $\sqrt{-1}\int_{\partial\Delta} \bar f \frac{df}{dz} dz > 0$.

Exercise 1.13. Let f be a holomorphic function on $\mathbb{C}$ such that $|f|(z) < |P(z)|$ for some polynomial $P(z)$. Prove that f is polynomial.

Exercise 1.14. Let f_i be a collection of holomorphic functions on a disk such that $\sum_{i=0}^\infty |f_i(z)|$ conferges uniformly on Δ. Prove that $\sum_{i=0}^\infty |f_i'(z)|$ converges uniformly on Δ.

Exercise 1.15. Let f be a bounded holomorphic function on $\{z \in \mathbb{C} \ \mid \ \operatorname{Re}(z) > 0\}$. Suppose that $f(n) = 0$ for all $n \in \mathbb{Z}^{>0}$. Prove that $f = 0$.

2 Homogeneous spaces

Exercise 2.1. Construct a fibration with total space $SU(3)$, base S^5 and fiber S^3.

Exercise 2.2. Prove that $\mathbb{C}P^n$ admits an $U(n+1)$-invariant Riemannian metric. Prove that this metric is unique up to a constant multiplier.

Exercise 2.3. Consider an $U(3)$-invariant symmetric 3-form $\eta \in \mathsf{Sym}^3 T^*\mathbb{C}P^2$. Prove that $\eta = 0$.

Exercise 2.4. Consider the space $\operatorname{AdS}(2) := SO^+(1,2)/SO^+(1,1)$ (so-called "2-dimensional anti-de Sitter space"). Prove that $\operatorname{AdS}(2)$ does not admit an $SO^+(1,2)$-invariant Riemannian structure.

Exercise 2.5. Prove that $\operatorname{AdS}(2) = SO^+(1,2)/SO^+(1,1)$ is not simply connected.

Exercise 2.6. Let V be an n-dimensional Hermitian complex space. Prove that the space $B \subset \mathbb{P}_{\mathbb{C}}V$ of all positive complex lines in V is biholomorphic to a ball in $\mathbb{C}^{n-1}$.

Exercise 2.7. Let V be an odd-dimensional vector space equipped with a positive definite quadratic form. Prove that the center of $SO(V)$ is trivial.

Exercise 2.8 (2 points). Let $V = \mathbb{R}^n$ be a Euclidean vector space, and $\operatorname{Gr}(2,V) = \frac{SO(n)}{SO(2)\times SO(n-2)}$ the Grassmannian of 2-planes in V. Prove that $\operatorname{Gr}(2,V)$ admits an $SO(n)$-invariant almost complex structure.

Exercise 2.9. Let X be the disk Δ with Poincare metric, and S^1X the space of all vectors of length 1 in $T\Delta$. Prove that the action of $SO(1,2)$ on S^1X is free and transitive.

Definition 2.1. **Horocycle** on a Poincaré plane is an orbit of a parabolic subgroup $P_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in PSL(2,\mathbb{R}) = SO^+(1,2)$.

Exercise 2.10. Prove that the group of isometries $\operatorname{Iso}(\mathbb{H}^2) = SO^+(1,2)$ acts transitively on the set of all horocycles.

Exercise 2.11. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $p_1, p_2 \in V$ two vectors with positive square, and $\tilde{p}_1, \tilde{p}_2$ the points in $\mathbb{P}V^+ = \mathbb{H}^2$ corresponding to p_1, p_2. Define **tance** of p_1, p_2 as $\mathsf{ta}(p_1, p_2) := \frac{q(p_1,p_2)^2}{q(p_1,p_1)q(p_2,p_2)}$. Prove that the distance between points $\tilde{p}_1, \tilde{p}_2$ in hyperbolic metric can be expressed through $\mathsf{ta}(p_1, p_2)$.

3 Poincaré metric and the automorphism groups

Exercise 3.1. Let M be a compact metric space, and $\operatorname{Iso}(M)$ the group of its isometries. We equip $\operatorname{Iso}(M)$ with topology of uniform convergence. Prove that $\operatorname{Iso}(M)$ is compact.

Exercise 3.2. Let M be a compact, Kobayashi-hyperbolic complex manifold. Prove that M does not admit non-zero holomorphic vector fields.

Exercise 3.3 (2 points). Let $M = \Delta/\Gamma$ be a compact Riemann surface, uniformized by a disk Δ. Prove that Γ has no fixed points on the absolute.

Exercise 3.4. Let Γ be an infinite cyclic subgroup of $SO^+(1,2)$, and $\delta \in SO^+(1,2)$ an element commuting with Γ. Let S be the set of fixed point of the action of Γ on the absolute. Prove that $\delta\Big|_S = \mathsf{Id}$.

Definition 3.1. Let l be a geodesic on a Poincaré plane. **A symmetry with an axis** l is an isometry changing orientation and acting trivially on l.

Exercise 3.5. Let l_1, l_2 be two geodesics in $\mathbb{H}^2$, and s_1, s_2 the corresponding symmetries. Prove that $s_1 s_2$ is parabolic if l_1, l_2 have a common end in the absolute. Prove that $s_1 s_2$ is hyperbolic if l_1, l_2 do not intersect in $\bar{\mathbb{H}}^2$.

Exercise 3.6. Let $u \in SO^+(1,2)$ be an isometry of $\mathbb{H}^2$ such that $u(x) \neq x$, but $u(u(x)) = x$. Prove that $u(u(y)) = y$ for all $y \in \mathbb{H}^2$.

Exercise 3.7. Let T be an absolute triangle, l_1, l_2, l_2 its edges, and s_1, s_2, s_3 the corresponding symmetries. Prove that $T \cap s_1 s_2 s_3(T) = \emptyset$.

Exercise 3.8 (3 points). Let Δ be a disk in $\mathbb{C}$ and M a compact complex manifold which is Kobayashi hyperbolic. Prove that any holomorphic map $\Psi: \ \Delta\backslash 0 \longrightarrow M$ can be holomorphically extended to Δ.

Definition 3.2. **A circle** with center x and radius r on a Poincaré plane $\mathbb{H}^2$ is the set $\{y \in \mathbb{H}^2 \ \mid \ d(x,y) = r\}$

Exercise 3.9. Let S be a circle on a disk $\Delta \subset \mathbb{C}$ with Poincaré metric. Prove that S is a circle in Euclidean geometry on $\Delta \subset \mathbb{C}$.

Exercise 3.10. Let r be a maximal radius of a circle in Poincaré plane which can be inscribed in a triangle. Prove that $r < \infty$.

Exercise 3.11. Let γ_1, γ_2 be two geodesics in Poincaré plane with the same end in the absolute. Prove that for each $\varepsilon > 0$ there exists $x \in \gamma_1, y \in \gamma_2$ such that $d(x,y) < \varepsilon$.

Exercise 3.12. Let $a, b, c \in \mathrm{Abs}$ be three distinct points on the absolute, and $A \in SO^+(1,2)$ an isometry of Poincaré plane which fixes a, b, c. Prove that $A = \mathsf{Id}$.

Exercise 3.13 (2 points). Let T be a triangle on a Poincare plane with vertices (a, b, c) and angle $\angle abc = \frac{\pi}{2}$. Prove that there exists a number C independent from a, b, c such that $|ac| + C > |ab| + |bc|$

Exercise 3.14 (2 points). Let $I_1, I_2 \in SL(2, \mathbb{R})$ be two operators satisfying $I_1^2 = I_2^2 = -\mathsf{Id}$. Prove that $|\operatorname{Tr} I_1 I_2| \geqslant 2$, with equality if and only if $I_1 = \pm I_2$.

4 Coverings, fundamental group, topology

Exercise 4.1. Let $V = \mathbb{R}^3$ be a vector space with quadratic form of signature (1,2), and $\mathbb{P}V^- \subset \mathbb{P}V = \mathbb{R}P^3$ the space of negative lines. Prove that $\mathbb{P}V^-$ is homeomorphic to a Möbius strip.

Exercise 4.2. Let M be a Riemann surface with infinite fundamental group. Prove that any continuous map $S^2 \longrightarrow M$ is homotopic to a trivial map (map to a point).

Exercise 4.3. Let M be a manifold with infinite fundamental group, $\tilde{M}$ its universal covering, and $\tilde{M} \times_M \tilde{M}$ the fibered product. Prove that $\tilde{M} \times_M \tilde{M}$ has infinitely many connected components.

Exercise 4.4. Let $\tilde{M} \longrightarrow M$ be a connected covering, and $G = \operatorname{Aut}_M(\tilde{M})$ a group of automorphisms of the covering. Prove that the action of G on $\tilde{M}$ is free, and the quotient space $\tilde{M}/G$ is also a covering of M.

Definition 4.1. **Free group** is a fundamental group of a bouquet of circles (a collection of circles glued in one point).

Exercise 4.5. Let $M_1 = S^1 \times S^1$ be a torus and M be M_1 without a point. Prove that its fundamental group is free.

Exercise 4.6. Let M be a simply connected manifold. Prove that any real rank 1 bundle on M is trivial.

Exercise 4.7. Prove that all real vector bundles on $\mathbb{R}$ are trivial. Construct a non-trivial vector bundle on S^1 or prove it does not exist.

Exercise 4.8. Let $TS^2 \oplus \mathbb{R}$ be a direct sum of a tangent bundle TS^2 and a trivial 1-dimensional bundle. Is the bundle $TS^2 \oplus \mathbb{R}$ trivial?

Exercise 4.9. Let $B \subset TM$ be a 2-dimensional sub-bundle, and

$$\Phi : \Lambda^2 B \longrightarrow TM/B$$

its Frobenius form. Find $B \subset T\mathbb{R}^3$ such that Φ nowhere vanishes.

Exercise 4.10. Find a rank 1 sub-bundle $B \subset TS^3$ such that the corresponding foliation has non-compact leaves.