Complex manifolds of dimension 1

lecture 2

Misha Verbitsky

IMPA, sala 232

January 8, 2020

Almost complex manifolds (reminder)

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: Let (V, I) be a space equipped with a complex structure $I: V \longrightarrow V$, $I^2 = -\operatorname{Id}$. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Complex manifolds and almost complex manifolds (reminder)

DEFINITION: **Standard almost complex structure** is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi: (M,I) \longrightarrow (N,J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: **A complex manifold** is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above.

Conversely, to determine an almost complex structure on M it suffices to define the Hodge decomposition $\Lambda^1(M) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$, but $\Lambda^{1,0}(M)$ **is generated by differentials of holomorphic functions,** and $\Lambda^{0,1}(M)$ is its complex conjugate. ∎

THEOREM: Let (M, I) be an almost complex manifold, $\dim_{\mathbb{R}} M = 2$. **Then I is integrable.**

Proof: Later in this course. ∎

Riemannian manifolds

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any path $\gamma : [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x - y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x,y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x,y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x,y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y,x)$.

DEFINITION: A skew-symmetric form $\omega(x,y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

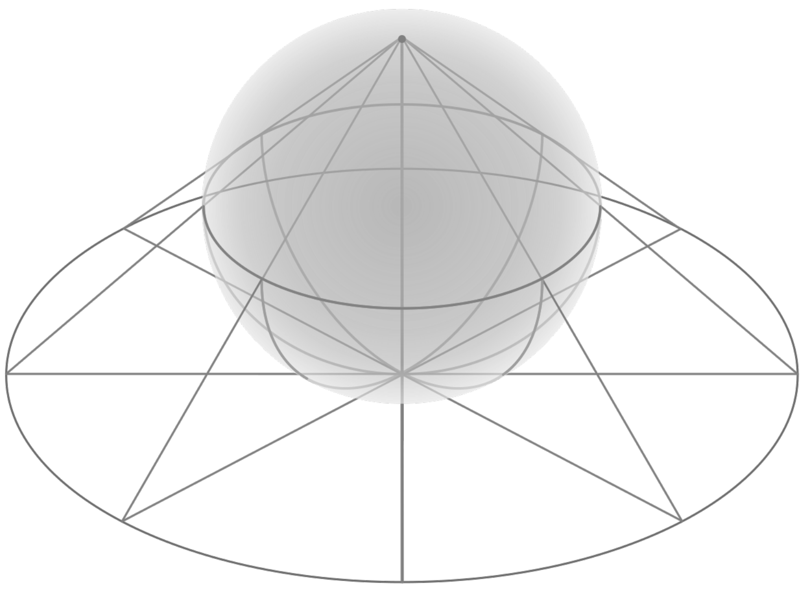
DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

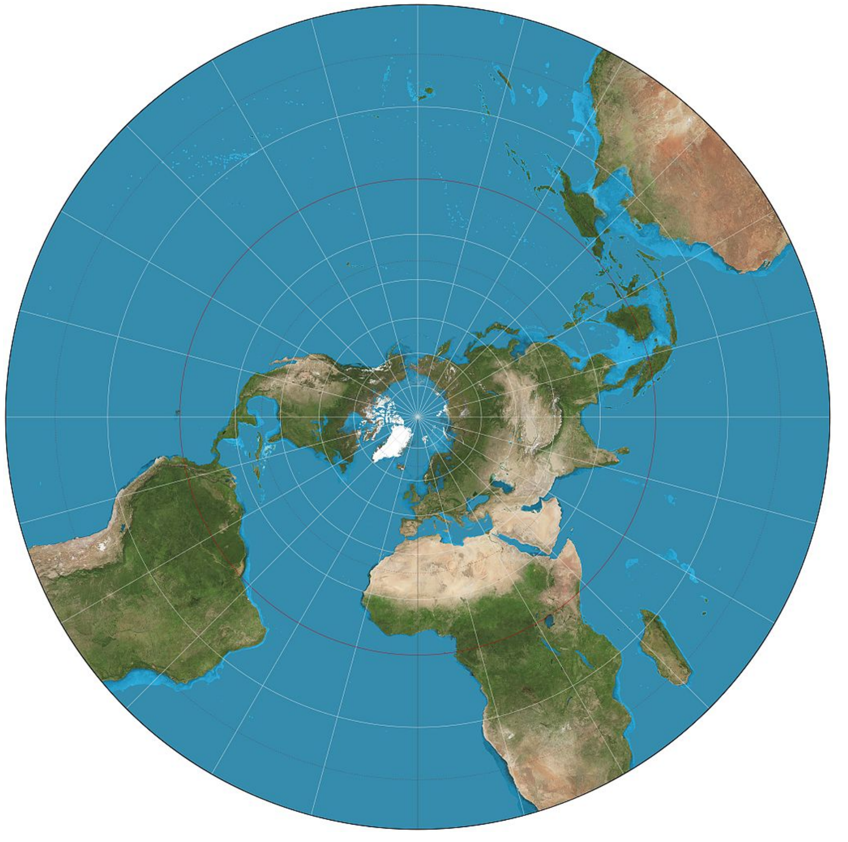
CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

To prove that any two Hermitian metrics are conformally equivalent, **we need to consider the standard $U(1)$-action on a complex vector space** (see the next slide).

Stereographic projection

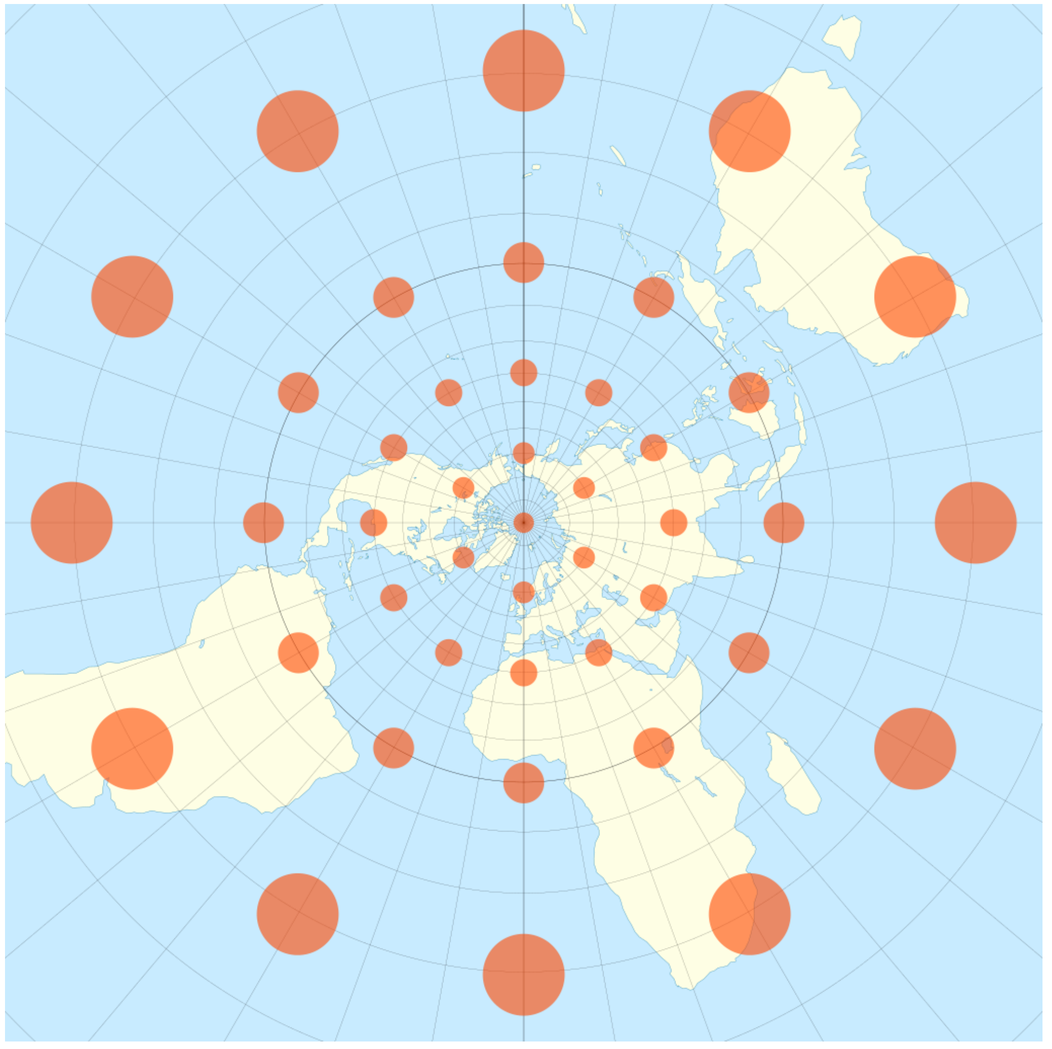




Stereographic projection is a light projection from the south pole to a plane tangent to the north pole.

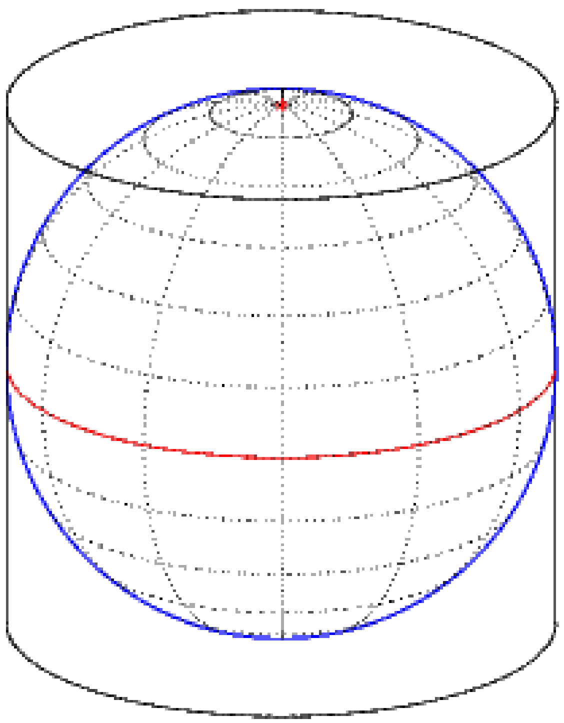
stereographic projection is conformal (prove it!)

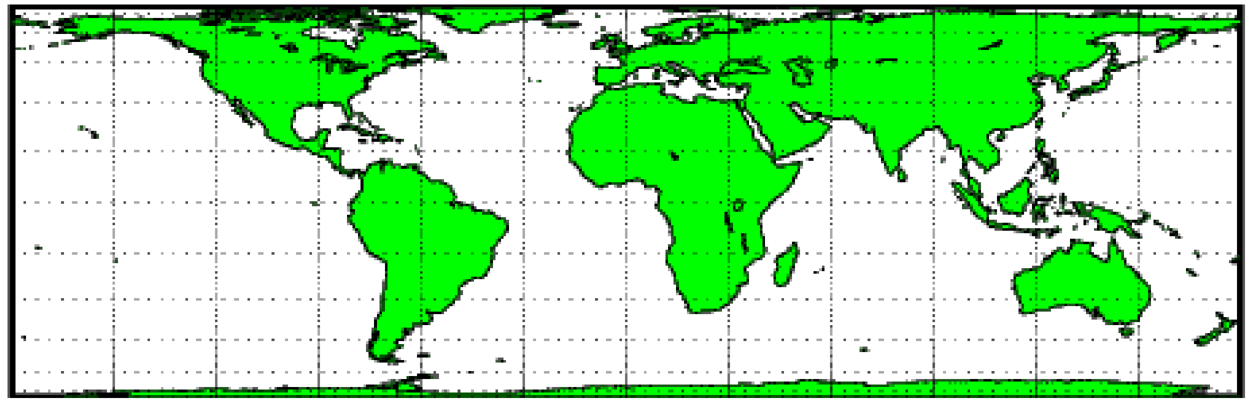
Stereographic projection (2)



The stereographic projection with Tissot's indicatrix of deformation.

Cylindrical projection





Cylindrical projection is not conformal. However, it is volume-preserving.

Standard $U(1)$-action

DEFINITION: Let (V, I) be a real vector space equipped with a complex structure, $U(1)$ the group of unit complex numbers, $U(1) = e^{\sqrt{-1}\,\pi t}$, $t \in \mathbb{R}$. Define the action of $U(1)$ on V as follows: $\rho(t) = e^{tI}$. This is called **the standard $U(1)$-action on a complex vector space**. To prove that this formula defines an action if $U(1) = \mathbb{R}/2\pi\mathbb{Z}$, it suffices to show that $e^{2\pi I} = 1$, which is clear from the eigenvalue decomposition of I.

CLAIM: Let (V, I, h) be a Hermitian vector space, and $\rho: U(1) \longrightarrow GL(V)$ the standard $U(1)$-action. **Then h is $U(1)$-invariant.**

Proof: It suffices to show that $\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = 0$. However, $\frac{d}{dt} e^{tI}(x)\Big|_{t=t_0} = I(e^{t_0 I}(x))$, hence

$$\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = h(I(\rho(t)x), \rho(t)x) + h(\rho(t)x, I(\rho(t)x)) = 2\omega(x, x) = 0.$$

■

Hermitian metrics in $\dim_{\mathbb{R}} = 2$.

COROLLARY: Let h, h' be Hermitian metrics on a space (V, I) of real dimension 2. **Then h and h' are proportional.**

Proof: h and h' are constant on any $U(1)$-orbit. Multiplying h' by a constant, we may assume that $h = h'$ on a $U(1)$-orbit $U(1)x$. Then $h = h'$ everywhere, because **for each non-zero vector $v \in V$, $tv \in U(1)x$ for some $t \in \mathbb{R}$, giving $h(v, v) = t^{-2}h(tv, tv) = t^{-2}h'(tv, tv) = h'(v, v)$.** ■

DEFINITION: Given two Hermitian forms h, h' on (V, I), with $\dim_{\mathbb{R}} V = 2$, we denote by $\frac{h'}{h}$ a constant t such that $h' = th$.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**.

Proof: $h' = \frac{h'}{h} h$. ■

EXERCISE: Prove that **Riemannian structure on M is uniquely defined by its conformal class and its Riemannian volume form**.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

EXERCISE: Prove that **a continuous map from one Riemannian surface to another is holomorphic if and only if it preserves the conformal structure** everywhere.

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has** $M = G/H$, where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\operatorname{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\operatorname{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\operatorname{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \operatorname{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: **Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM,** for any $x \in M$.

■

Space forms

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined in the next slide.

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■