Complex manifolds of dimension 1

lecture 3

Misha Verbitsky

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Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \ \mid \ g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\operatorname{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\operatorname{St}_x(G)$ on T_xM.

DEFINITION: A bilinear symmetric form (or any tensor) Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant bilinear symmetric form (or any tensor) on T_xM, where $M = G/H$ is a homogeneous space. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g^*(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \operatorname{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Let $M = G/H$ be a homogeneous space and $x \in M$ a point. Then the G-invariant bilinear forms (or tensors) on $M = G/H$ **are in bijective correspondence with isotropy invariant bilinear forms (tensors)** on the vector space T_xM. ■

Space forms

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of affine isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined in the next slide.

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique up to a constant multiplier G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof. Step 1: Let g, g' be two metrics. Clearly, it suffices to show that the functions $x \longrightarrow g(x)$ and $x \longrightarrow g'(x)$ are proportional. Fix a vector v on a unit sphere. Replacing g' by $\frac{g(v)}{g'(v)} g'$ if necessary, we can assume that $g = g'$ on a sphere. Indeed, a sphere is an orbit of $SO(n)$, and g, g' are $SO(n)$-invariant.

Step 2: **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx for appropriate $x \in S^{n-1}$, $\lambda \in \mathbb{R}$. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

Hermitian and conformal structures (reminder)

DEFINITION: Let $h \in \text{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: A Riemannian metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1}\,)$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

Poincaré-Koebe uniformization theorem

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: **(Poincaré-Koebe uniformization theorem)** Let M be a Riemann surface. **Then M admits a complete metric of constant curvature in the same conformal class.**

COROLLARY: **Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.**

COROLLARY: **Any simply connected Riemann surface is conformally equivalent to a space form.**

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Matrix exponent and Lie groups

DEFINITION: Exponent of an endomorphism A is $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$. **Logarithm** of an endomorphism $1 + A$ is $\log(1 + A) := \sum_{n=1}^{\infty} -(-1)^n \frac{A^n}{n}$

EXERCISE: Prove that **exponent is inverse to logarithm in a neighbourhood of 0.**

EXERCISE: Prove that **if $A, B \in \operatorname{End}(V)$ commute, one has $e^{A+B} = e^A e^B$.**

EXERCISE: Find an example when $A, B \in \operatorname{End}(V)$ **do not commute, and $e^{A+B} \neq e^A e^B$.**

EXERCISE: Prove that **exponent is invertible in a sufficiently small neighbourhood of 0** (use the inverse map theorem).

DEFINITION: Let $W \subset \operatorname{End}(V)$ be a subspace obtained by logarithms of all elements in a neighbourhood of zero of a subgroup $G \subset GL(V)$. A group $G \subset GL(V)$ is called **a Lie subgroup of $GL(V)$**, or **a matrix Lie group**, if it is closed and equal to e^W in a neighbourhood of unity. In this case W is called its **Lie algebra**.

REMARK: It is possible to show that **any closed subgroup of $GL(V)$ is a matrix group**. However, for many practical purposes this can be assumed.

Lie groups: first examples

EXAMPLE: From (local) invertibility of exponent it follows that in a neighbourhood of Id_V we have $GL(V) = e^W$, for $W = \operatorname{End}(V)$ **(prove it)**.

EXERCISE: Prove that $\det e^A = e^{\operatorname{Tr} A}$**,** where $\operatorname{Tr} A$ is a trace of A.

EXAMPLE: Let $SL(V)$ be the group of all matrices with determinant 1, and $\operatorname{End}_0(V)$ the space of all matrices with trace 0. Then $e^{\operatorname{End}_0(V)} = SL(V)$ **(prove it).** This implies that $SL(V)$ **is also a Lie group.**

Lie groups as submanifolds

DEFINITION: A subset $M \subset \mathbb{R}^n$ is **an m-dimensional smooth submanifold** if for each $x \in M$ there exists an open in $\mathbb{R}^n$ neighbourhood $U \ni x$ and a diffeomorphism from U to an open ball $B \subset \mathbb{R}^n$ which maps $U \cap M$ to an intersection $B \cap R^m$ of B and an m-dimensional linear subspace.

PROPOSITION: Let $G \subset \operatorname{End}(V)$ be a matrix subgroup in $GL(V)$. **Then G is a submanifold.**

Proof. Step 1: From inverse function theorem, it follows that $A \longrightarrow e^A$ is a diffeomorphism on a neighbourhood of 0 mapping the Lie algebra W of G to G.

Step 2: For any $g \in G$, consider the map $x \longrightarrow ge^x$. This map defines a diffeomorphism between a neighbourhood of 0 in $\operatorname{End}(V)$ and a neighbourhood gU of g, mapping W to $gU \subset G$. ■

Orthogonal group as a Lie group

DEFINITION: Let V be a vector space equipped with a non-degenerate bilinear symmetric form h. Then the group of all endomorphisms of V preserving h and orientation is called **(special) orthogonal group**, denoted by $SO(V,h)$.

DEFINITION: Consider the space of all $A \in \operatorname{End}(V)$ which satisfy $h(Ax,y) = -h(x,Ay)$. This space is called **the space of antisymmetric matrices** and denoted $\mathfrak{so}(V,h)$.

REMARK: Clearly, $\mathfrak{so}(V,h) = \{A \in \operatorname{End}(V) \ \ | \ \ A^t = -A\}$.

THEOREM: ***$SO(V,h)$ is a Lie group, and $\mathfrak{so}(V,h)$ its Lie algebra.***

Proof. Step 1:

$$0 = \frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}w).$$

If h is e^{tA}-invariant, this gives $0 = h(Av,w) + h(v,Aw)$, hence A is antisymmetric.

Orthogonal group as a Lie group (2)

THEOREM: $SO(V,h)$ **is a Lie group, and** $\mathfrak{so}(V,h)$ **its Lie algebra.**

Proof. Step 1:

$$0 = \frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(e^{tA}A(v), e^{tA}w) + h(e^{tA}v, e^{tA}A(w)).$$

If h is e^{tA}-invariant, this gives $0 = h(Av,w) + h(v,Aw)$, hence A is antisymmetric.

Step 2: Conversely, suppose that A is antisymmetric. Then

$$\frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}Aw) = 0,$$

hence $h(e^{tA}v, e^{tA}w)$ is independent from t and equal to $h(v,w)$. ∎

Classical Lie groups

EXERCISE: Prove that the following groups are Lie groups.

$U(n)$ **("unitary group")**: the group of complex linear automorphisms of $\mathbb{C}^n$ preserving a Hermitian form.

$SU(n)$: **("special unitary group")**: the group of complex linear automorphisms of $\mathbb{C}^n$ of determinant 1 preserving a Hermitian form.

$Sp(2n, \mathbb{R})$ **("symplectic group")**: the group of linear automorphisms of $\mathbb{R}^{2n}$ preserving a non-degenerate, antisymmetric 2-form.

Properties of matrix groups

LEMMA: Let $G \subset GL(V)$ be a matrix Lie group, equal to e^W in a neighbourhood of 1. **Then $W = T_e G \subset \operatorname{End}(V) = T_e GL(V)$.**

Proof: The exponent map $W \longrightarrow e^W \subset G$ is an isomorphism in a neighbourhood of 0, but **the differential of this map is identity.**

LEMMA: Let G be a connected Lie group. **Then G is generated by any neighbourhood of unity.**

Proof: A subgroup $H \subset G$ generated by a given neighbourhood of unity $U \ni e$ is open, The map $U \longrightarrow G$ mapping (u, x) to ux is a diffeomorphism from U to a neighbourhood of x hence it is open. Since any orbit Hx of H acting on G is open, it is also closed, and (unless G is disconnected) there is only one such orbit. ■

Surjective homomorphisms of matrix groups

COROLLARY 1: Let $\psi : G \longrightarrow G'$ be a Lie group homomorphism. Suppose that its differential is surjective. **Then Ψ is surjective on a connected component of unity.**

Proof: Let $W = T_e G$ and $W' = T_e G'$. Since the differential of Ψ is surjective, Ψ is surjective to some neighbourhood of unity by the inverse function theorem. However, a neighbourhood of unity generates G' by the previous lemma. Therefore, Ψ is surjective. ■

COROLLARY 2: Let $\psi : G \longrightarrow G'$ be a Lie group homomorphism. Assume that ψ is injective in a neighbourhood of unity, and $\dim G = \dim G'$. **Then ψ is surjective on a connected component of unity.**

Proof: The differential of ψ is an isomorphism (it is an injective map of vector spaces of the same dimension). Now ψ is surjective by Corollary 1. ■

Group of unitary quaternions

DEFINITION: A quaternion z is called **unitary** if $|z|^2 := z\overline{z} = 1$. The group of unitary quaternions is denoted by $U(1,\mathbb{H})$. **This is a group of all quaternions satisfying $z^{-1} = \overline{z}$.**

CLAIM: Let $\operatorname{im}\mathbb{H} := \mathbb{R}^3$ be the space $aI+bJ+cK$ of all imaginary quaternions. The map $x, y \longrightarrow -\operatorname{Re}(xy)$ defines scalar product on $\operatorname{im}\mathbb{H}$.

CLAIM: This scalar product is positive definite.

Proof: Indeed, if $z = aI + bJ + cK$, $\operatorname{Re}(z^2) = -a^2 - b^2 - c^2$. ■

COROLLARY: Consider the action of $U(1,\mathbb{H})$ on $\operatorname{Im} H$ with $h \in U(1,\mathbb{H})$ mapping $z \in \operatorname{Im}\mathbb{H}$ to $hz\overline{h}$. Since $\overline{hz\overline{h}} = h\overline{z}\overline{h}$, this quaternion also imaginary. Also, $|hz\overline{h}|^2 = hz\overline{h}h\overline{z}\overline{h} = h|z|^2\overline{h} = |z|^2$. **This implies that $U(1,\mathbb{H})$ acts on the space $\operatorname{im}\mathbb{H}$ by isometries.**

DEFINITION: Denote the group of all oriented linear isometries of $\mathbb{R}^3$ by $SO(3)$. This group is called **the group of rotations of $\mathbb{R}^3$.**

REMARK: We have just defined a group homomorphism $U(1,\mathbb{H}) \longrightarrow SO(3)$ mapping h, z to $hz\overline{h}$.

Group of rotations of $\mathbb{R}^3$

Similar to complex numbers which can be used to describe rotations of $\mathbb{R}^2$, quaternions can be used to describe rotations of $\mathbb{R}^3$.

THEOREM: Let $U(1,\mathbb{H})$ be the group of unitary quaternions acting on $\mathbb{R}^3 = \operatorname{Im} H$ as above: $h(x) := hx\overline{h}$. Then **the corresponding group homomorphism defines an isomorphism $\Psi : U(1,\mathbb{H})/\{\pm 1\} \xrightarrow{\sim} SO(3)$.**

Proof. Step 1: First, any quaternion h which lies in the kernel of the homomorpism $U(1,\mathbb{H}) \longrightarrow SO(3)$ commmutes with all imaginary quaternions, Such a quaternion must be real **(check this).** Since $|h| = 1$, we have $h = \pm 1$. **This implies that Ψ is injective.**

Step 2: These groups are 3-dimensional. **Then Ψ is surjective by Corollary 2.**

COROLLARY: The group $SO(3)$ is identified with the real projective space $\mathbb{R}P^3$.

Proof: Indeed, $U(1,\mathbb{H})$ is identified with a 3-sphere, and $\mathbb{R}P^3 := S^3/\{\pm 1\}$.■

The group $SO(4)$

Consider the following scalar product on $\mathbb{H} = \mathbb{R}^4$: $g(x,y) = \operatorname{Re}(x\overline{y})$. Clearly, it is positive definite. Let $U(1,\mathbb{H})\times U(1,\mathbb{H})$ act on $\mathbb{H}$ as follows: $h_1, h_2, z \longrightarrow h_1 z\overline{h}_2$, with $z \in \mathbb{H}$ and $h_1, h_2 \in U(1,\mathbb{H})$. Clearly, $|h_1 z\overline{h}_2|^2 = h_1 z\overline{h}_2 h_2 \overline{z}\overline{h}_1 = h_1 z\overline{z}\overline{h}_1 = z\overline{z}$, hence **the group $U(1,\mathbb{H}) \times U(1,\mathbb{H})$ acts on $\mathbb{H} = \mathbb{R}^4$ by isometries.** Clearly, $\ker \Psi$ contains a pair $(-1,-1) \subset U(1,\mathbb{H}) \times U(1,\mathbb{H})$. We denote the group generated by $(-1,-1)$ as $\{\pm 1\} \subset U(1,\mathbb{H}) \times U(1,\mathbb{H})$.

THEOREM: Denote by $SO(4)$ the group of linear orthogonal automorphisms of $\mathbb{R}^4$, and let $\Psi: U(1,\mathbb{H}) \times U(1,\mathbb{H})/\{\pm 1\} \longrightarrow SO(4)$ be the group homomorphism constructed above, $h_1, h_2(x) = h_1 x\overline{h}_2$. **Then Ψ is an isomorphism**. In particular, **$SO(4)$ is diffeomorphic to $S^3 \times S^3/\{\pm 1\}$.**

Proof. Step 1: Again, let $(h_1,h_2) \in \ker\Psi$. Since $\Psi(h_1,h_2)(1) = 1$, this gives $h_2 = \overline{h}_1 = h_1^{-1}$. However, $h_1 z h_1^{-1} = z$ means that h_1 commutes with z, which implies that h_1 commutes with all quaternions, hence it is real. Then $h_1 = \pm 1$. **This proves injectivity of Ψ.**

Step 2: The group $SO(4)$ is 6-dimensional **(prove it)**, and $U(1,\mathbb{H}) \times U(1,\mathbb{H})$ is also 6-dimensional. Then Ψ is surjective by Corollary 2. ■