Complex manifolds of dimension 1

lecture 5: 1-dimensional Lie groups

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Left-invariant vector fields

REMARK: A group acts on itself in three different ways: there is **left action** $g(x) = gx$, **right action** $g(x) = xg^{-1}$, and **adjoint action** $g(x) = gxg^{-1}$,

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields.

REMARK: Since the group acts on itself freely and transitively, left-invariant vector fields on G are identified T_eG. Indeed, **any vector $x \in T_eG$ can be extended to a left-invariant vector field in a unique way.**

REMARK: The same is true for any left-invariant tensor on G: it can be obtained in a unique way from a tensor on a vector space T_eG.

Lie algebra

REMARK: Since **the commutator of left-invariant vector fields is left-invariant,** commutator is well defined on the space of left invariant vector fields A. Commutator is a bilinear, antisymmetric operation $A \times A \longrightarrow A$ which satisfies **the Jacobi identity:**

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

DEFINITION: A **Lie algebra** is a vector space A equipped with a bilinear, antisymmetric operation $A \times A \longrightarrow A$ which satisfies the Jacobi identity.

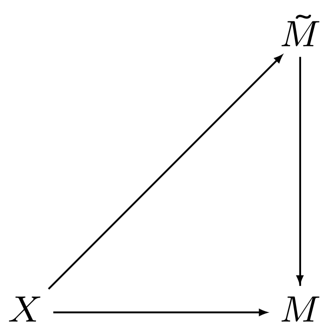
THEOREM: (The main theorem of Lie theory) A simply connected Lie group **is uniquely determined by its Lie algebra.** Every finite-dimensional Lie algebra **is obtained as a Lie algebra of a simply connected Lie group**.

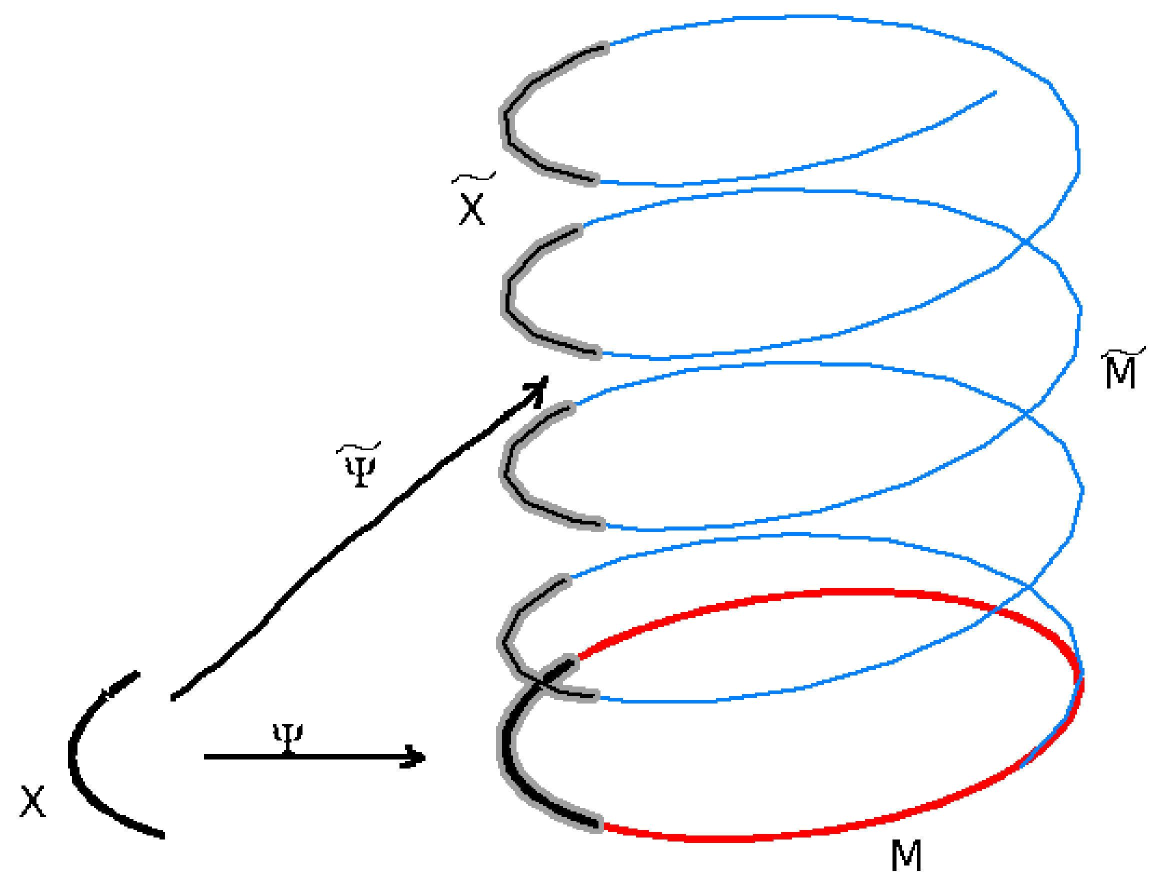
DEFINITION: Adjoint representation of a Lie group is the action of G on its Lie algebra T_eG obtained from the adjoint action of G on itself, $g(x) = gxg^{-1}$.

REMARK: Any matrix Lie group $G \subset GL(V)$, **is generated by exponents of its Lie algebra** $\text{Lie}(G)$**,** and locally in a neighbourhood of zero the exponent map $\exp: \text{Lie}(G) \longrightarrow G$ is a diffeomorphism.

Homotopy lifting principle

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.

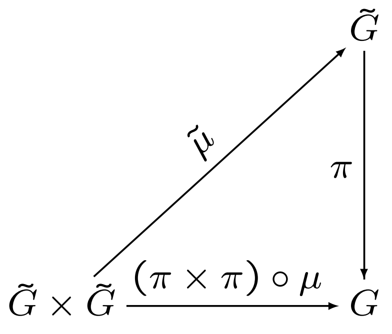


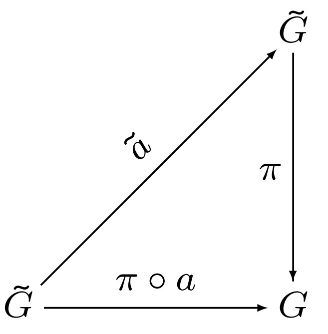


Universal covering of a Lie group

THEOREM: Let G be a connected Lie group, and $\tilde G$ its universal covering. **Then $\tilde G$ has a unique structure of a Lie group, such that the covering map $\pi: \tilde G \longrightarrow G$ is a homomorphism.**

Proof: The multiplication map $\tilde G \longrightarrow \tilde G \xrightarrow{\tilde\mu} \tilde G$ is a lifting of the composition of π and multiplication $\tilde G \times \tilde G \xrightarrow{\pi\times\pi} G\times G \xrightarrow{\mu} G$ mapping the unity $\tilde e \times \tilde e$ to $\tilde e$. Similarly, the inverse map $\tilde a: \tilde G \longrightarrow \tilde G$ is a lifting of the inverse $a: G \longrightarrow G$ mapping $\tilde e$ to $\tilde e$





Uniqueness and group identities on $\tilde G$ both follow from the uniqueness of the homotopy lifting. ■

Classification of 1-dimensional Lie groups

Exercise 1: Prove that **any non-trivial discrete subgroup of $\mathbb{R}$ is cyclic** (isomorphic to $\mathbb{Z}$).

THEOREM: Any 1-dimensional connected Lie group G is isomorphic to S^1 or $\mathbb{R}$.

Proof. Step 1: Any 1-dimensional manifold is diffeomorphic to S^1 or $\mathbb{R}$. By Exercise 1 it suffices to prove that any simply connected, connected 1-dimensional Lie group is isomorphic to $\mathbb{R}$.

Classification of 1-dimensional Lie groups (2)

THEOREM: Any 1-dimensional connected Lie group G is isomorphic to S^1 or $\mathbb{R}$.

Step 2: Since G is simply connected, it is diffeomorphic to $\mathbb{R}$. Let $v \in T_e G$ be a non-zero tangent vector, $\vec{v} \in TG$ the corresponding left-invariant vector field, and $E: \mathbb{R} \longrightarrow G$ a solution of the ODE

$$\frac{d}{dt} E(t) = \vec{v}. \quad (*)$$

mapping 0 to e. A solution of (*), considered as a map from $\mathbb{R}$ to $G = \mathbb{R}$, exists and is uniquely determined by $P(0)$ by the uniqueness and existence of solutions of ODE. Since the left action L_g of G on itself preserves $\vec{v}$, it maps solutions of (*) to solutions of (*). Let $g = E(s)$. Then $t \longrightarrow L_{g^{-1}} E(s+t)$ is a solution of (*) which maps 0 to $E(s)^{-1} E(s) = e$, hence $L_{g^{-1}} E(s+t) = E(t)$ and $E(s+t) = E(s)E(t)$. Therefore, the map $E: \mathbb{R} \longrightarrow G$ is a group homomorphism.

Step 3: Differential of E is non-degenerate, hence E is locally a diffeomorphism; since G is connected, G is generated by a neighbourhood of 0, hence E is surjective. If E is not injective, its kernel is discrete, but then $\ker E = \mathbb{Z}$, and G is a circle. Therefore, E is invertible. ■

Group of unitary quaternions (reminder)

DEFINITION: A quaternion z is called **unitary** if $|z|^2 := z\bar{z} = 1$. The group of unitary quaternions is denoted by $U(1, \mathbb{H})$. **This is a group of all quaternions satisfying $z^{-1} = \bar{z}$.**

CLAIM: Let $\operatorname{im} \mathbb{H} := \mathbb{R}^3$ be the space $aI+bJ+cK$ of all imaginary quaternions. The map $x, y \longrightarrow -\operatorname{Re}(xy)$ defines scalar product on $\operatorname{im} \mathbb{H}$.

CLAIM: This scalar product is positive definite.

Proof: Indeed, if $z = aI + bJ + cK$, $\operatorname{Re}(z^2) = -a^2 - b^2 - c^2$. ■

COROLLARY: The group $U(1, \mathbb{H})$ acts on the space $\operatorname{im} \mathbb{H}$ by isometries.

REMARK: We have just defined a group homomorphism $U(1, \mathbb{H}) \longrightarrow SO(3)$ mapping h, z to $hz\bar{h}$.

COROLLARY 2 (Lecture 3): Let $\psi : G \longrightarrow G'$ be a Lie group homomorphism. Assume that ψ is injective in a neighbourhood of unity, and $\dim G = \dim G'$. **Then ψ is surjective on a connected component of unity.**

Group of rotations of $\mathbb{R}^3$ (reminder)

Similar to complex numbers which can be used to describe rotations of $\mathbb{R}^2$, quaternions can be used to describe rotations of $\mathbb{R}^3$.

THEOREM: Let $U(1, \mathbb{H})$ be the group of unitary quaternions acting on $\mathbb{R}^3 = \operatorname{Im} H$ as above: $h(x) := hx\overline{h}$. Then **the corresponding group homomorphism defines an isomorphism $\Psi : U(1, \mathbb{H})/\{\pm 1\} \tilde{\longrightarrow} SO(3)$.**

Proof. Step 1: First, any quaternion h which lies in the kernel of the homomorpism $U(1, \mathbb{H}) \longrightarrow SO(3)$ commmutes with all imaginary quaternions, Such a quaternion must be real **(check this).** Since $|h| = 1$, we have $h = \pm 1$. **This implies that Ψ is injective.**

Step 2: The groups $U(1, \mathbb{H})$ and $SO(3)$ are 3-dimensional. **Then Ψ is surjective by Corollary 2.**

COROLLARY: The group $SO(3)$ is identified with the real projective space $\mathbb{R}P^3$.

Proof: Indeed, $U(1, \mathbb{H})$ is identified with a 3-sphere, and $\mathbb{R}P^3 := S^3/\{\pm 1\}$.■

$U(\mathbb{H}, 1)$ is generated by exponents

LEMMA: The group $U(\mathbb{H}, 1)$ is generated locally by exponents of imaginary quaternions.

Proof: Let h be an imaginary quaternion. Then $\frac{d}{dt}(e^{th}, e^{th}) = (he^{th}, e^{th}) + (e^{th}, he^{th}) = 0$ because $(h(x), y) = -(x, h(y))$ for any imaginary quaternion. Indeed, rescaling h if necessary, we may assume that $h^2 = -1$, then $(h(x), y) = (h^2 x, hy) = -(x, hy)$. ■

$SU(2) = U(\mathbb{H}, 1)$

The left action of $U(\mathbb{H}, 1)$ on $\mathbb{H} = \mathbb{C}^2$ commutes with the right action of the algebra $\mathbb{C}$ on $\mathbb{H} = \mathbb{C}^2$. This defines a homomorphism $U(\mathbb{H}, 1) \longrightarrow U(2)$.

THEOREM: This homomorphism **defines an isomorphism $U(\mathbb{H}, 1) \cong SU(2)$,** where $SU(2) \subset U(2)$ is a subgroup of **special unitary matrices** (unitary matrices with determinant 1).

Proof. Step 1: The group $U(2)$ is 4-dimensional, because it is a fixed point set of an anti-complex involution $A \longrightarrow (A^t)^{-1}$ in a space $GL(2, \mathbb{C})$ of real dimension 8. The group $SU(2)$ is a kernel of the determinant map $U(2) \xrightarrow{\det} U(1)$, **hence it is 3-dimensional.**

Step 2: The map $U(\mathbb{H}, 1) \longrightarrow U(2)$ is by construction injective. Its image is generated by exponents of imaginary quaternions. The elements of $\operatorname{im} \mathbb{H}$ act on $\mathbb{H} = \mathbb{C}^2$ by traceless matrices **(prove this)**. Using the formula $e^{\operatorname{Tr} A} = \det e^A$, we obtain that their exponents have trivial determinant. **This gives an injective map $U(\mathbb{H}, 1) \longrightarrow SU(2)$.** It is surjective by Corollary 2. ■

Complex projective space (reminder)

DEFINITION: Let $V = \mathbb{C}^n$ be a complex vector space equipped with a Hermitian form h, and $U(n)$ the group of complex endomorphisms of V preserving h. This group is called **the complex isometry group**.

DEFINITION: **Complex projective space** $\mathbb{C}P^n$ is the space of 1-dimensional subspaces (lines) in $\mathbb{C}^{n+1}$.

REMARK: Since the group $U(n+1)$ of unitary matrices acts on lines in $\mathbb{C}^{n+1}$ transitively **(prove it)**, $\mathbb{C}P^n$ **is a homogeneous space,** $\mathbb{C}P^n = \frac{U(n+1)}{U(1)\times U(n)}$**,** where $U(1) \times U(n)$ is a stabilizer of a line in $\mathbb{C}^{n+1}$.

EXAMPLE: $\mathbb{C}P^1$ is S^2.

Homogeneous and affine coordinates on $\mathbb{C}P^n$ (reminder)

DEFINITION: We identify $\mathbb{C}P^n$ with the set of $n+1$-tuples $x_0 : x_1 : ... : x_n$ defined up to equivalence $x_0 : x_1 : ... : x_n \sim \lambda x_0 : \lambda x_1 : ... : \lambda x_n$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** in the chart $x_k \neq 0$ are are $\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}$. The space $\mathbb{C}P^n$ is a union of $n+1$ affine charts identified with $\mathbb{C}^n$, with the complement to each chart identified with $\mathbb{C}P^{n-1}$.

CLAIM: Complex projective space is a complex manifold, with the atlas given by affine charts $\mathbb{A}_k = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}\right\}$, and the transition functions mapping the set

$$\mathbb{A}_k \cap \mathbb{A}_l = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k} \;\middle|\; x_l \neq 0\right\}$$

to

$$\mathbb{A}_l \cap \mathbb{A}_k = \left\{\frac{x_0}{x_l} : \frac{x_1}{x_l} : ... : 1 : ... : \frac{x_n}{x_l} \;\middle|\; x_k \neq 0\right\}$$

as a multiplication of all terms by the scalar $\frac{x_k}{x_l}$.

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined by the following lemma, proven in Lecture 3.

LEMMA: Let $M = G/H$ be a simply connected space form. **Then M admits a unique up to a constant multiplier G-invariant Riemannian form.**

REMARK: We shall consider space forms as Riemannian manifolds equipped with a G-invariant Riemannian form.

Next subject: We are going to **classify conformal automorphisms of all space forms.**

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.

The following theorem was proven in Lecture 4.

THEOREM: The natural map from $PGL(2, \mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

$PU(2) = SO(3)$

DEFINITION: Let $U(2) \subset GL(2,\mathbb{C})$ be the group of unitary matrices, and $PU(2)$ its quotient by the group $U(1)$ of diagonal matrices. It is called **projective unitary group**.

REMARK: $PU(2)$ is a quotient of $SU(2)$ by its center $-\operatorname{Id}$ **(prove it)**. The group $U(2)$ acts on $\mathbb{C}P^1$, its action is factorized through $PU(2)$, and all non-trivial $g \in PU(2)$ act on $\mathbb{C}P^1$ non-trivially **(prove it)**.

THEOREM: **$PU(2)$ is isomorphic to $SO(3)$,** the isotropy group of its action on $\mathbb{C}P^1$ is $U(1)$, and the $U(2)$-invariant metric on $\mathbb{C}P^1$ is isometric to the standard Riemannian metric on S^2.

Proof: As shown above, $PU(2) = \frac{SU(2)}{\pm 1}$, and $SO(3) = \frac{U(1,\mathbb{H})}{\pm 1}$. On the other hand, $SU(2) = U(1,\mathbb{H})$.

An element $a \in SU(2)$ fixing a line $x \in \mathbb{C}P^1$ acts on its orthogonal complement by rotations. Since $\det a = 1$, the angle of this rotation uniquely determines the angle of rotation of a on the line x. Therefore, the isotropy group of $SU(2)$-action on $\mathbb{C}P^1$ is S^1. For $PU(2)$ it is $S^1/\{\pm 1\} = S^1$.

Finally, there exists only one, up to a constant, $SO(3) = PU(2)$-invariant metric on $\frac{SO(3)}{SO(2)} = S^2$. ■