Complex manifolds of dimension 1

lecture 6: Pseudo-Hermitian forms, circles on a sphere and Möbius group

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Hermitian and pseudo-Hermitian forms

DEFINITION: Let (V, I) be a (real) vector space equipped with a complex structure, and h a bilinear symmetric form. It is called **pseudo-Hermitian** if $h(x, y) = h(Ix, Iy)$.

REMARK: The corresponding quadratic form $x \mapsto h(x, x)$ is sometimes writen as $h(x)$. **One can recover $h(x, y)$ from $h(x)$ as usual: $2h(x, y) = h(x + y) - h(x) - h(y)$.**

REMARK: Often one considers a complex-valued form $h(x, y) + \sqrt{-1}h(x, Iy)$. It is **sesquilinear** as a form on the complex space: $h(\lambda x, y) = \lambda(x, y)$, $h(x, \lambda y) = \overline{\lambda}(x, y)$, for any $\lambda \in \mathbb{C}$, and the imaginary part $\sqrt{-1}\, h(x, Iy)$ is anti-symmetric.

CLAIM: Let (V, I, h) be a pseudo-Hermitian vector space. Consider V as a complex vector space, $\dim_{\mathbb{C}} V = n$. Then there exists a basis $z_1, ..., z_n$ in V such that $h(z_i, z_j) = 0$ for $i \neq j$ (such a basis is called **orthogonal**). Moreover, this basis can be chosen in such a way that $h(z_i, z_i)$ is ± 1 or 0 (such a basis is called **orthonormal**).

Orthonormal basis for a pseudo-Hermitian form

CLAIM: For any pseudo-Hermitian form h on (V, I), **there exists orthonormal basis $z_1, ..., z_n$.**

Proof: Use induction on $\dim V$. If $h = 0$, this claim is clear. Assume that $h \neq 0$. For any $A \subset V$, denote by $A^\perp$ the space $\{x \in V \mid h(x,a) = 0 \forall a \in A\}$.

Choose any $z_1 \in V$ such that $h(z_1, z_1) \neq 0$, and let $z_1^{\perp,\mathbb{C}} := \langle z_1, I(z_1)\rangle^\perp = z_1^\perp \cap I(z_1)^\perp$. This is a complex vector space which is orthogonal to z_1. It can also be obtained as an orthogonal complement with respect to the sesquilinear form $h(x,y) + \sqrt{-1}\, h(x, Iy)$.

By induction assumption, the space $z_1^{\perp,\mathbb{C}}$ has an orthonormal basis $z_2, ..., z_n$. **Then $z_1, ..., z_n$ is an orthogonal basis in V.** Replacing z_1 by $h(z_1, z_1)^{1/2} z_1$, we obtain an orthonormal basis $z_1, ..., z_n$. ■

Signature of a Hermitian form

REMARK: By Sylvester's law of inertia, the number of z_i such that $h(z_i, z_i) = 1$, $h(z_i, z_i) = -1$ and $h(z_i, z_i) = 0$ **is independent form the choice of an orthonormal basis.**

DEFINITION: Let (V, I, h) be a vector space with non-degenerate Hermitian form, and $z_1, ..., z_n$ an orthonormal basis, $h(z_i, z_i) = 1$ for $i = 1, ...p$ and $h(z_i, z_i) = 1$ for $i = p + 1, ..., n$, with $q = n - p$. Then h is called **Hermitian form of signature (p, q)**. The group of complex linear automorphisms preserving h **is denoted $U(p, q)$.**

Normal form for a pair of Hermitian forms

Theorem 1: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

Theorem 1': Let $V = \mathbb{C}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two (pseudo-)Hermitian forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

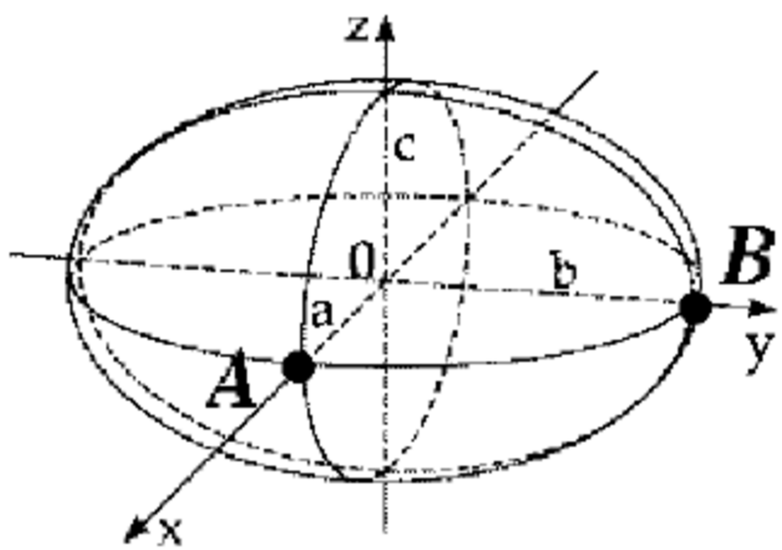
REMARK: In this basis, h' is written as diagonal matrix, with eigenvalues $\alpha_1, ..., \alpha_n$ independent from the choice of the basis. Indeed, consider h, h' as maps from V to V^*, $h(v) = h(v, \cdot)$. Then $h_1 h^{-1}$ is an endomorphism with eigenvalues $\alpha_1, ..., \alpha_n$. **This implies that Theorem 1 gives a normal form of the pair h, h'.**

Finding principal axes of an ellipsoid

REMARK: Theorem 1 implies the following statement about ellipsoids: for any positive definite quadratic form q in $\mathbb{R}^n$, consider the ellipsoid

$$S = \{v \in V \mid q(v) = 1\}.$$

The group $SO(n)$ acts on $\mathbb{R}^n$ preserving the standard scalar product. **Then for some $g \in SO(n)$, $g(S)$ is given by equation $\sum a_i x_i^2 = 1$, where $a_i > 0$.** This is called **finding principal axes of an ellipsoid**.



Compactness

DEFINITION: Recall that a subset $Z \subset \mathbb{R}^n$ is called **(sequentially) compact** if any sequence $x_1, ..., x_n, ... \subset Z$ has a converging subsequence.

THEOREM: A subset $Z \subset \mathbb{R}^n$ is **sequentially compact if and only if Z is closed and bounded** (that is, contained in a ball of finite diameter).

EXERCISE: Let f be a continuous function on a compact Z. **Prove that Z is bounded and attains its supremum on Z.**

COROLLARY: Let f be a continuous function on a sphere $S^n \subset \mathbb{R}^{n+1}$. **Then f is bounded, and attains its supremum.**

Further on, we prove the following lemma.

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complement with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

Maximum of a quadratic form on a sphere

Further on, we prove the following lemma.

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complement with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

This lemma immediately implies Theorem 1. Let h, h', x as above. Using induction, we may assume that $x^{\perp_h} = x^{\perp_{h'}}$ admits a basis $x_2, ..., x_n$ which is orthonormal for h and orthogonal for h'. **Then $x, x_2, ..., x_n$ is a basis we need.**

Similarly one proves Theorem 1'. Take $x \in S$ as above. Then $I(x)$ is also a maximum for q'. The orthogonal complements to $x, I(x)$ with respect to h and h' coincide by our lemma. Therefore, $W = \langle x, Ix\rangle^{\perp_h} = \langle x, Ix\rangle^{\perp_{h'}}$. We obtain a complex vector space W orthogonal to x with respect to h and h'. Using induction, we find a basis $x_2, ..., x_n$ in W which is orthonormal for h and orthogonal for h'. **Then $x, x_2, ..., x_n$ is such a basis in V.**

Maximum of a quadratic form on a sphere

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complements with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

Proof: Let us rescale q, q' in such a way that $q \geqslant q'$, with equality on x. Suppose that $v \in x^{\perp_h}$. Then $q(x + \varepsilon v) = q(x) + \varepsilon^2 q(v)$. However, $q'(x + \varepsilon v) = q(x) + \varepsilon^2 q'(v) + 2\varepsilon h'(v, x)$. This gives

$$q(x) + \varepsilon^2 q(v) \geqslant q(x) + \varepsilon^2 q'(v) + 2\varepsilon h'(v, x)$$

cancelling $q(x)$ and dividing by $\varepsilon > 0$, obtain

$$\varepsilon(q(v) - q'(v)) \geqslant 2h'(v, x).$$

for all $\varepsilon > 0$. This implies that $0 \geqslant 2h'(v, x)$ for all $v \in x^{\perp_h}$. **Since $v \mapsto h'(v, x)$ is a linear form on v, inequality $0 \geqslant h'(v, x)$ implies that $h'(v, x) = 0$.** ■

Möbius transforms (reminder)

DEFINITION: **Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.

The following theorem was proven in Lecture 4.

THEOREM: The natural map from $PGL(2, \mathbb{C})$ to the group $\mathrm{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

$PU(2) = SO(3)$ (reminder)

DEFINITION: Let $U(2) \subset GL(2, \mathbb{C})$ be the group of unitary matrices, and $PU(2)$ its quotient by the group $U(1)$ of diagonal matrices. It is called **projective unitary group**.

REMARK: $PU(2)$ is a quotient of $SU(2)$ by its center $-\operatorname{Id}$ **(prove it)**. The group $U(2)$ acts on $\mathbb{C}P^1$, its action is factorized through $PU(2)$, and all non-trivial $g \in PU(2)$ act on $\mathbb{C}P^1$ non-trivially **(prove it)**.

THEOREM: **$PU(2)$ is isomorphic to $SO(3)$,** the isotropy group of its action on $\mathbb{C}P^1$ is $U(1)$, and the $U(2)$-invariant metric on $\mathbb{C}P^1$ is isometric to the standard Riemannian metric on S^2.

Circles on a sphere

DEFINITION: A circle in S^2 is an orbit of rotation subgroup, that is, a subgroup $U \subset SO(3) = PU(2) \subset PGL(2,\mathbb{C})$ isomorphic to S^1 and acting on $S^2 = \mathbb{C}P^1$ by isometries.

REMARK: Let U be a rotation group rotating S^2 around an axis passing through x and $y \in S^2$. **Any orbit C of U satisfies $d(x,v) = const$ for all $v \in C$.**

LEMMA: Let z_1, z_2 be a basis in $V = \mathbb{C}^2$, and $h(az_1 + bz_2) = \alpha|a|^2 - \beta|b|^2$ a pseudo-Hermitian form, with $\alpha, \beta \geqslant 0$. **Then the set $Z_h = \mathbb{P}\{x \in V \ \mid\ h(x) = 0\}$ is a circle in $\mathbb{C}P^1$, and all circles can be obtained this way.**

Proof: In homogeneous coordinates, Z_h is the set of all $x : y$ such that $\alpha|x|^2 = \beta|y|^2$, and rotation acts as $x : y \longrightarrow x : e^{\sqrt{-1}\,\theta} y$. Clearly, the orbits of rotation are precisely the sets Z_h for different α, β. ∎

Properties of Möbius transform

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof. Step 1: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). There exists an orthonormal basis $z_1, z_2 \in V$ such that $h(az_1 + bz_2) = \alpha|a|^2 - \beta|b|^2$ with $\alpha, \beta > 0$ real numbers. The set $\{z \ | \ h(z,z) = 0\}$ is invariant under the rotation $z_1, z_2 \longrightarrow e^{-\sqrt{-1}\,\theta}z_1, e^{\sqrt{-1}\,\theta}z_2$, hence **it is a circle.**

Step 2: By the previous lemma, all circles are obtained this way.

Step 3: $PGL(2,\mathbb{C})$ maps pseudo-Hermitian forms to pseudo-Hermitian forms of the same signature, and therefore **preserves circles.** ■

Orbits of compact one-parametric subgroups in $PGL(2, \mathbb{C})$

LEMMA: Let $G \cong S^1$ be a compact 1-dimensional subgroup in $PGL(2, \mathbb{C})$. **Then any G-orbit in $\mathbb{C}P^1$ is a circle.**

Proof: Let $V = \mathbb{C}^2$, and consider the natural projection map

$$\pi : \ SL(V) \longrightarrow PGL(2, \mathbb{C}) = SL(V)/\pm 1.$$

Then $\tilde{G} = \pi^{-1}(G)$ is compact. Chose a $\tilde{G}$-invariant Hermitian metric h on V by averaging a given metric with $\tilde{G}$-action. By definition, circles on $\mathbb{C}P^1$ are orbits of rotation subgroups in $SU(V, h)$. **Since $u(\tilde{G})$ is a 1-dimensional compact subgroup in $SU(V, h)$, its orbit is a circle.** ■