Complex manifolds of dimension 1

lecture 7: Isometries of Poincaré plane

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Some low-dimensional Lie group isomorphisms

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in $GL(\text{Lie}(G))$ with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the center Z of G (prove it).

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional real space equipped with a scalar product of signature (1,2), $SO^+(1,2)$ a connected component of unity, and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$, $SO^+(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, with $\kappa(a,b) := \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism ("Corollary 2" in Lecture 3). ■

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

THEOREM: The natural map from $PGL(2, \mathbb{C})$ to the group of Möbius transforms is an isomorphism.

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2, \mathbb{C})$.

PROPOSITION: The action of $PGL(2, \mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

Proof: By Cauchy formula, $f(0) = \frac{1}{2\pi} \int_{\partial\Delta} f(z) \frac{dz}{-\sqrt{-1}\, z}$, where Δ is a disk in $\mathbb{C}$. An elementary calculation gives $\frac{dz}{-\sqrt{-1}\, z}\Big|_{\partial\Delta} = \operatorname{Vol}(\partial\Delta)$ – the volume form on $\partial\Delta$. Therefore, $f(0)$ is the average of $f(z)$ on the circle, and it is the average of $f(z)$ on the disk Δ. Now, absolute value of the average $|\operatorname{Av}_{x\in S}\mu(x)|$ of a complex-valued function μ on a set S is equal to $\max_{x\in S}|\mu(s)|$ only if $\mu = \text{const}$ almost everywhere on S **(check this)**. ■

LEMMA: (Schwartz lemma) Let $f : \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = \text{const}$. ■

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of $PGL(2,\mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi : \ G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\mathsf{St}_x(G_1) = \mathsf{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker \Psi$ belongs to $\mathsf{St}_x(G_1) = \mathsf{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \mathsf{St}_x(G_1) = \mathsf{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ \mid \ h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi: \ PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma **(prove it!)**. ■

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta, \text{flat metric})$.**

Proof: The group $\operatorname{Aut}(\Delta) = PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique up to a contant multiplier **(prove it)**. ■

Upper half-plane

REMARK: The map $z \longrightarrow -(z-\sqrt{-1}\,)^{-1} - \frac{\sqrt{-1}}{2}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}^2$ **(prove it)**.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}^2$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}^2$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

COROLLARY: The group of conformal automorphisms of $\mathbb{H}^2$ acts on $\mathbb{H}^2$ preserving a unique, up to a constant, Riemannian metric. **The Riemannian manifold $PSL(2,\mathbb{R})/S^1$ obtained this way is isometric to a hyperbolic space.**

Upper half-plane as a Riemannian manifold

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}^2$. **Then the Riemannian structure s on $\mathbb{H}^2$ is written as** $s = \text{const}\, \frac{dx^2 + dy^2}{y^2}$.

Proof: Since the complex structure on $\mathbb{H}^2$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2 + dy^2)$, where $\mu \in C^\infty(\mathbb{H}^2)$. **It remains to find μ, using the fact that s is $PSL(2, \mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $z \longrightarrow z + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the homothety $H_\lambda(z) = \lambda z$ also fixes s; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2(dx^2 + dy^2)$, we have $\mu(\lambda z) = \lambda^{-2}\mu(z)$ for any $z \in \mathbb{H}^2$. The only function $\mu(x, y)$ which is constant in x and satisfies $\mu(\lambda y) = \lambda^{-2}\mu(y)$ is $\mu(x, y) = \text{const}\, y^{-2}$. ■

Geodesics on Riemannian manifold

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.

Geodesics in Poincaré half-plane

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}^2$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and l the vertical line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}^2$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}^2$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line: $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}^2) = PSL(2,\mathbb{R})$** ■

Geodesics in Poincaré half-plane

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

Poincaré metric on a disk

DEFINITION: **Poincaré metric** on the unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier; prove it)**.

DEFINITION: Let $f : M \longrightarrow M_1$ be a map of metric spaces. Then f is called C-**Lipschitz** if $d(x, y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi : \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential, taken with respect to the Poincaré metric, satisfies $|D\varphi_x|_P \leqslant 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■