Complex manifolds of dimension 1

lecture 8 1/2: Isometries of the Poincaré plane (2)

Misha Verbitsky

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Reflections, geodesics, negative lines (reminder)

DEFINITION: A reflection on a hyperbolic plane is an involution which reverses orientation and has a fixed set of codimension 1.

EXAMPLE: Let the quadratic form q be written as $q(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2$. Then the map $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$ is clearly a reflection.

CLAIM: Fixed point set of a reflection is a geodesic. This produces **a bijection between the set of geodesics and the set of reflections.**

Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $\operatorname{Pos} := \{v \in V \ \ | \ \ q(v) > 0\}$, and $\mathbb{P}\operatorname{Pos}$ its projectivisation. Then $\mathbb{P}\operatorname{Pos} = SO^+(1,2)/SO(1)$ **(check this)**, giving $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$; **this is one of the standard models of a hyperbolic plane.**

REMARK: Let $l \subset V$ be **a line,** that is, a 1-dimensional subspace. The property $q(x,x) < 0$ for a non-zero $x \in l$ is written as $q(l,l) < 0$. A line l with $q(l,l) < 0$ is called **negative line**, a line with $q(l,l) > 0$ is called **positive line**.

PROPOSITION: Reflections on $\mathbb{P}\operatorname{Pos}$ **are in bijective correspondence with negative lines $l \subset V$.**

Geodesics and the absolute (reminder)

Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $\operatorname{Pos} := \{v \in V \mid q(v) > 0\}$, and $\mathbb{P}\operatorname{Pos}$ its projectivisation. Then $\mathbb{P}\operatorname{Pos} = SO^+(1,2)/SO(1)$, giving $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$.

DEFINITION: A line $l \in V$ is **isotropic** if $q(l,l) = 0$. **Absolute** of a hyperbolic plane $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$ is the set of all isotropic lines,

$$\operatorname{Abs} := \{l \in \mathbb{P}V \mid q(l,l) = 0\}.$$

It is identified with the boundary of the disk $\mathbb{P}\operatorname{Pos} \subset \mathbb{P}V = \mathbb{R}P^2\}$.

CLAIM: Let $l \in \mathbb{P}V$ be a negative line, and $\gamma := \mathbb{P}l^\perp \cap \mathbb{P}\operatorname{Pos}$ the corresponding geodesic. **Then $l^\perp$ intersects the absolute in precisely 2 points,** called **the boundary points of γ**, or **ends of γ**. Conversely, **every geodesic is uniquely determined by the two distinct points in the absolute.**

Proof: The plane $l^\perp$ has signature (1,1), and the set $q(v) = 0$ is a union of two isotropic lines in $l^\perp$. Each of these lines lies on the boundary of the set $\mathbb{P}l^\perp \cap \mathbb{P}\operatorname{Pos}$. Conversely, suppose that $\mu, \rho \in \operatorname{Abs}$ are two distinct lines. The corresponding 2-dimensional plane W has signature (1,1), because it has precisely two isotropic lines **(if it has more than two, $q|_W = 0$, which is impossible - prove it!)**. As shown above, $\mathbb{P}W \cap \mathbb{P}\operatorname{Pos}$ is a geodesic. ■

Isomorphism between $SO(1,2)$ and $PSL(2,\mathbb{R})$ (reminder)

CLAIM: $PSL(2,\mathbb{R}) \cong SO^+(1,2)$

Proof: Consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism ("Corollary 2" in Lecture 3). ■

PROPOSITION: Let V be a 2-dimensional vector space. **Then $\mathfrak{sl}(V)$ is isomorphic to** $\operatorname{Sym}^2(V)$ (the space of symmetric 2-tensors), and this isomorphism is compatible with the $SL(V)$-action.

Corollary 1: Let $A \in SL(2,\mathbb{R})$ be a matrix with eigenvalues α, α^{-1}, and $B \in SO(1,2)$ the endomorphism associated with A through $PSL(2,\mathbb{R}) \cong SO^+(1,2)$. **Then B has eigenvalues $\alpha^2, 1, \alpha^{-2}$.**

Classification of isometries of a hyperbolic plane (part 1)

THEOREM: Let $A \in SL(2, \mathbb{R})$, and $\alpha \in SO^+(1,2)$ the corresponding isometry of a hyperbolic plane. Denote by q the quadratic form of signature $(1,2)$ on R^3. Assume that $\alpha \neq \mathrm{Id}$, that is, $A \neq \pm 1$. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$. In this case α is called **"elliptic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| < 2$; it is conjugate to a rotation of a disk around 0.

(ii) α has an eigenvector x with $q(x,x) < 0$. In that case α is called **"hyperbolic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| > 2$; it is conjugate to a matrix $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, with $t \neq \pm 1$.

(iii) α has a unique eigenvector x with $q(x,x) = 0$. In that case α is called **"parabolic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| = 2$, and is conjugate to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

Iometries of a hyperbolic plane (elliptic case)

THEOREM: Let $A \in SL(2,\mathbb{R})$, and $\alpha \in SO^+(1,2)$ the corresponding isometry of a hyperbolic plane. Denote by q the quadratic form of signature $(1,2)$ on R^3. Assume that $\alpha \neq \mathrm{Id}$, that is, $A \neq \pm 1$. Then one and only one of these three cases occurs

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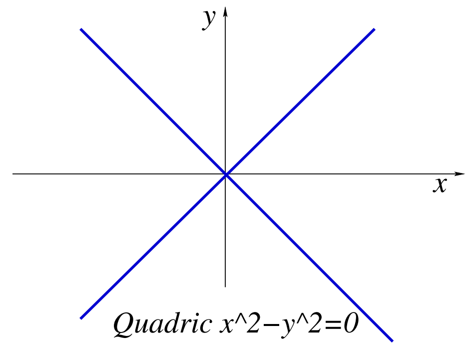
Isometries with a positive eigenvector. Let $u, v = u^{-1}$ be eigenvalues of A, and $u^2, v^2, 1$ eigenvalues of α (Corollary 1). The map α has a real eigenvector x. If $q(x,x) > 0$, α is an elliptic isometry. Then $\alpha(x) = x$ because $q(x,x) = q(\alpha(x), \alpha(x)) > 0$. The map α acts as rotation on $x^\perp$, which is 2-plane with negative definite scalar product. All subgroups $S^1 \subset SL(2,\mathbb{R})$ are conjugate to rotation (Lecture 7), hence $A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ and $|\operatorname{Tr} A| = 2|\cos t| < 2$. In this case $u = \overline{v} \in U(1)$.

Isometries of a hyperbolic plane (hyperbolic case)

Step 2: hyperbolic isometries.
If $q(x,x) < 0$, α is a hyperbolic isometry. In this case, α acts by isometry on a plane $x^\perp$ which has signature (1,1). The set Pos of vectors $z \in \mathbb{R}^3$ with positive square is $\{(a,b,c) \ | \ a^2 - b^2 - c^2 > 0\}$. Since $a \neq 0$, this set is disconnected. Since $\alpha \in SO^+(1,2)$, it preserves the connected components of Pos. Let $Q := \{v \in x^\perp \ | \ q(v,v) = 0\}$ be the corresponding homogeneous quadric in $x^\perp$. Clearly, **Q is a union of two lines. Since α preserves connected components of Pos, it acts on Q preserving the lines and the orientation.**



Let ρ, μ be non-collinear vectors generating these lines. The action of α on $\langle \rho, \mu \rangle$ is written by a matrix $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, with $t \in \mathbb{R}^{>0} \backslash \{1\}$. Then α is diagonalizable with eigenvalues $1, t, t^{-1}$, and A has eigenvalues $\pm\sqrt{t}, \pm\sqrt{t^{-1}}$. ■

REMARK: A hyperbolic isometry α fixes a unique geodesic with boundary in $\rho, \mu \in$ Abs. Indeed, α fixes two and only two points on Abs, and every geodesic is determined uniquely by two points on Abs.

Isometries of a hyperbolic plane (parabolic case)

Parabolic case: (iii) α has a unique eigenvector x with $q(x,x) = 0$. In that case α is called **"parabolic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| = 2$, and is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof: This occurs when α has no fixed points on $\mathbb{P}V \backslash \mathrm{Abs}$. In his case, α cannot fix two points $\rho, \mu \in \mathrm{Abs}$, because if it does, it fixes a 2-dimensional space $W = \langle \rho, \mu \rangle$, and then it fixes the line $W^\perp$ which is negative. This means that **α has a unique fixed point on $\mathbb{P}V$, which lies in Abs.** This implies that α has only one eigenvalue, which is equal to 1, and its Jordan normal form is

$$\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then A is not diagonalizable, which implies that its Jordan normal form is $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. ■

REMARK: All such matrices are conjugate, hence **a parabolic isometry is conjugate to the isometry of Poincaré plane given by $z \longrightarrow z + \lambda$, $\lambda \in \mathbb{R}$.**

Polygons in a hyperbolic plane

DEFINITION: Geodesic bisects a hyperbolic or euclidean plane onto two connected components, called **half-planes**. A **convex polygon** is an intersection of a (generally, finite) collection of half-planes. **A polygon** is a (generally, finite) union of convex polygons.

DEFINITION: Edge of a polygon is a connected interval of a geodesic obtained by intersection of the boundary ∂P of a polygon and a geodesic. **A vertex** of a polygon is an end of its edge, either in $\mathbb{H}^2$ or in Abs.

EXERCISE: Prove that **a convex polygon is uniquely determined by its vertices.**

EXERCISE: Let $P \subset \mathbb{H}^2$ be a convex polygon such that its closure in $\mathbb{H}^2 \cup$Abs has only finitely many points on Abs. Suppose that P has n vertices and $\alpha_1, ..., \alpha_k$ are interior angles for all vertices of P in $\mathbb{H}^2$. Prove that **there exists a constant $C > 0$ such that $\operatorname{Vol}(P) = (n-2)\pi - \sum \alpha_i$.**

Definition of a volume

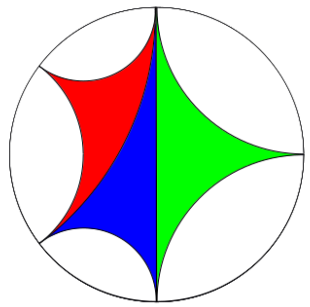
DEFINITION: Consider a function Φ from the set of all polygons on $\mathbb{H}^2$ to non-negative numbers. Assume that Φ is continuous as a function of vertices of a polygon and invariant under isometries. Assume that for any union $W = V_1 \cap V_2$ with V_1 intersecting V_2 only in the boundary, one has $\Phi(W) = \Phi(V_1) + \Phi(V_2)$ (the function is **additive**). Then Φ is called **a volume.**

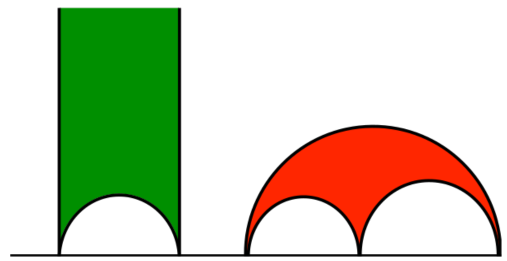
EXERCISE: Prove that **the volume is unique,** up to a constant multiplier.

EXERCISE: Prove the additivity of the function $\operatorname{Vol}(P) := (n-2)\pi - \sum \alpha_i$ **defined above.**

Ideal triangles

DEFINITION: An ideal triangle on a hyperbolic plane is a triangle with ver-





tices on Abs.

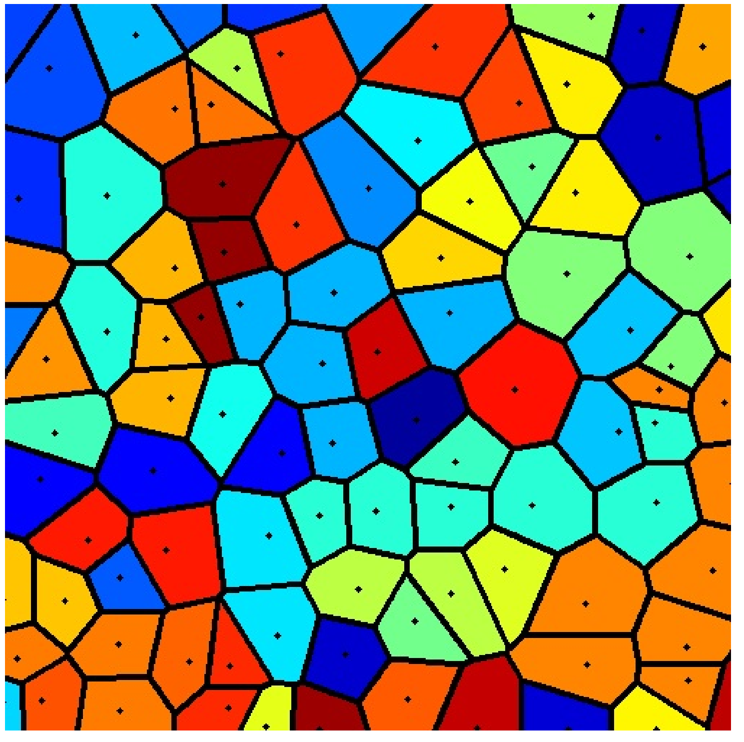
EXERCISE: Let $A\subset \mathbb{H}$ be an angle formed by intersection of two halh-planes. **Prove that A contains infinitely many ideal triangles.**

COROLLARY: Let P be a polygon which has finite volume. **Then $\partial P\cap$Abs is finite.**

EXERCISE: Prove it. ■

Voronoi partitions

DEFINITION: Let M be a metric space, and $S \subset M$ a finite subset. **Voronoi cell** associated with $x_i \in S$ is $\{z \in M \ \ | \ (z, x_i) \leqslant d(z, x_i) \forall j \neq i\}$. **Voronoi partition** is partition of M onto its Voronoi cells.



Voronoi partition

Fundamental domains and polygons

DEFINITION: Let Γ be a discrete group acting on a manifold M, and $U \subset M$ an open subset with piecewise smooth boundary. Assume that for any non-trivial $\gamma \in \Gamma$ one has $U \cap \gamma(U) = \emptyset$ and $\Gamma \cdot \overline{U} = M$, where $\overline{U}$ is closure of U. Then $\overline{U}$ is called **a fundamental domain** of the action of Γ.

THEOREM: Let Γ be a discrete group acting on a hyperbolic plane $\mathbb{H}^2$ by isometries. **Then Γ has a polyhedral fundamental domain P** with (possibly) finitely many vertices. If, moreover, $\mathbb{H}^2/\Gamma$ has finite volume, ∂P has at most finitely many points on Abs.

Proof: Clearly, $\operatorname{Vol}(P) = \operatorname{Vol}(\mathbb{H}^2/\Gamma)$. This takes care of the last assertion, because polygons with infinitely many points on Abs have infinite volume

To obtain P, take a point $s \in \mathbb{H}$, and let P be the Voronoi cell associated with the set $\Gamma \cdot s$. ■

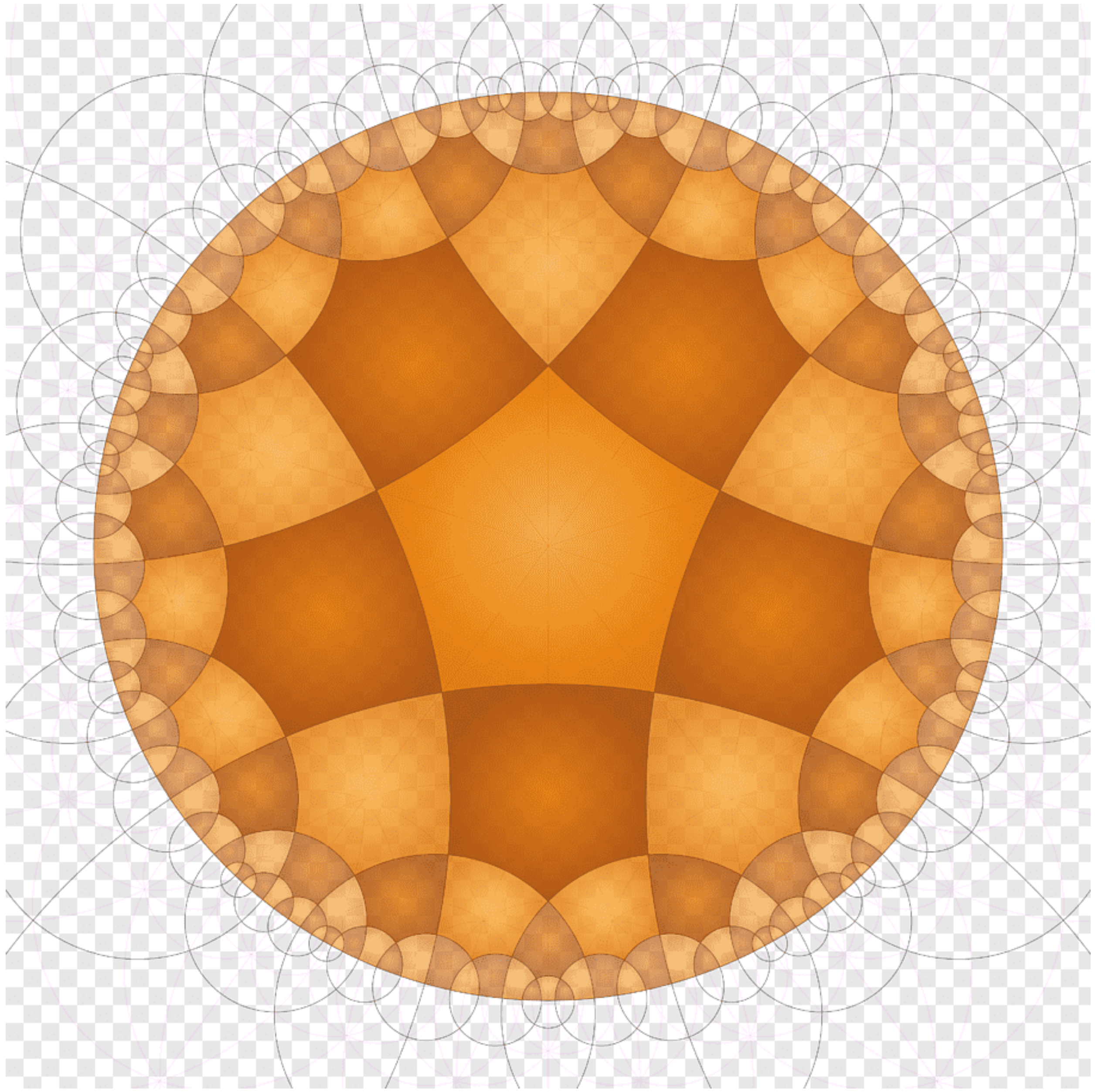
EXERCISE: Prove that in fact P **has finitely many vertices when** $\operatorname{Vol}(\mathbb{H}^2/\Gamma)$ **is finite.**

Semi-regular tilings

DEFINITION: A tiling of $\mathbb{H}^2$ is a partition of $\mathbb{H}^2$ onto polygons with finite volume. A tiling is **regular** if the group Γ of isometries preserving tilings acts transitively on vertices, edges and faces of the partition. A tiling T is **semi-regular** if Γ acts on the set of faces of T with finitely many orbits.

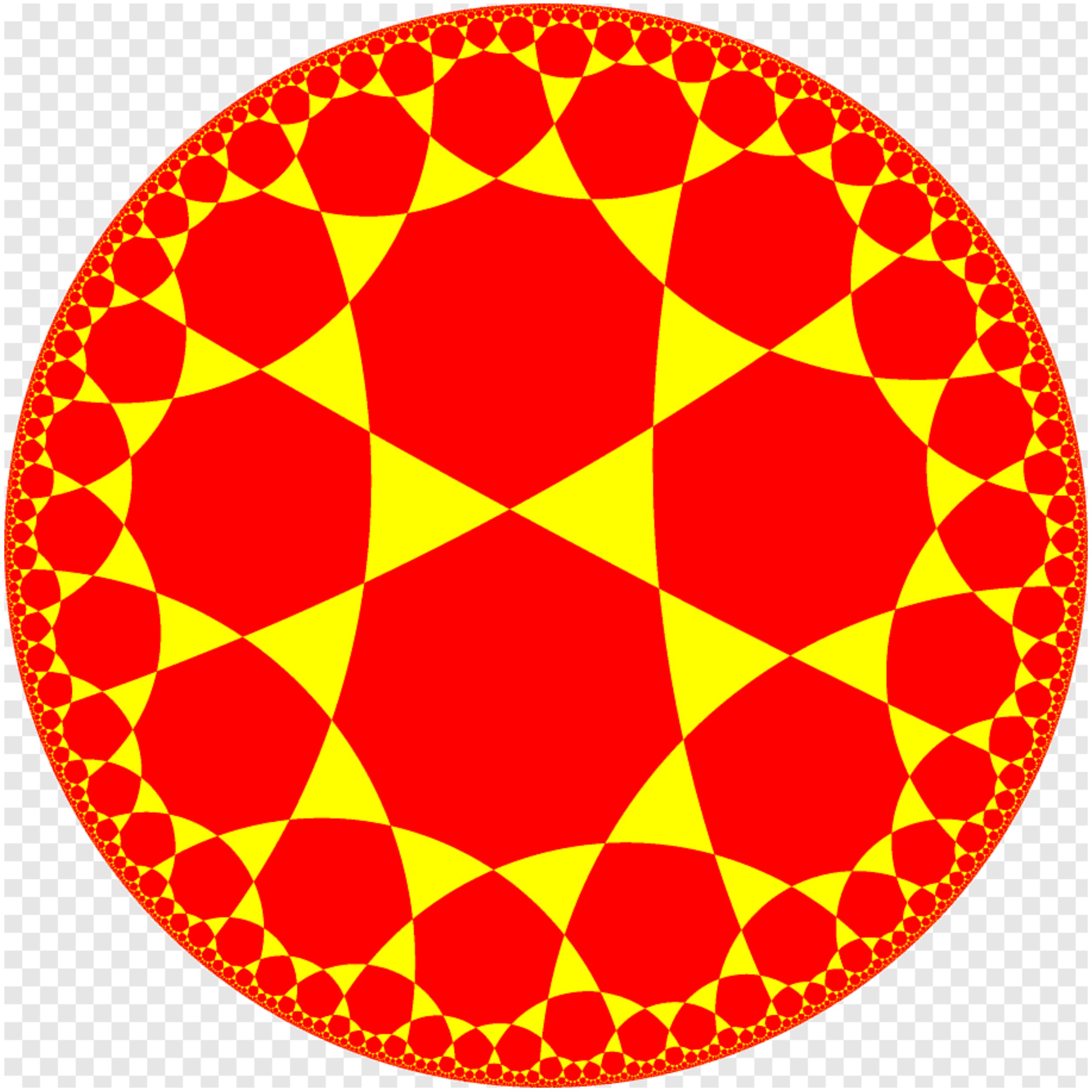
REMARK: Tilings is good a way to produce hyperbolic manifolds and Riemannian surfaces from a hyperbolic plane. Indeed, for any semi-regular tiling, T, **the quotient space $\mathbb{H}^2/\Gamma$ has finite volume.** Moreover, $\mathbb{H}^2/\Gamma$ **is compact if all polygons in T have no vertices in** Abs.

Regular tiling of $\mathbb{H}^2$ by right-angle pentagons



Regular tiling of $\mathbb{H}^2$ by right-angle pentagons

Semi-regular tiling of $\mathbb{H}^2$



Semi-regular tiling of $\mathbb{H}^2$ by octagons and triangles

Cocompact subgroups of $PSL(2, \mathbb{R})$ without torsion

DEFINITION: A discrete subgroup $\Gamma \subset PSL(2, \mathbb{R})$ is **cocompact** if $\mathbb{H}^2/\Gamma$ is compact.

THEOREM: (a part of Poincaré uniformization theorem)
Let S be a compact Riemannian surface of genus > 1. **Then $S = \mathbb{H}^2/\Gamma$ for $\Gamma \subset PSL(2, \mathbb{R})$ freely acting on $\mathbb{H}^2$.**

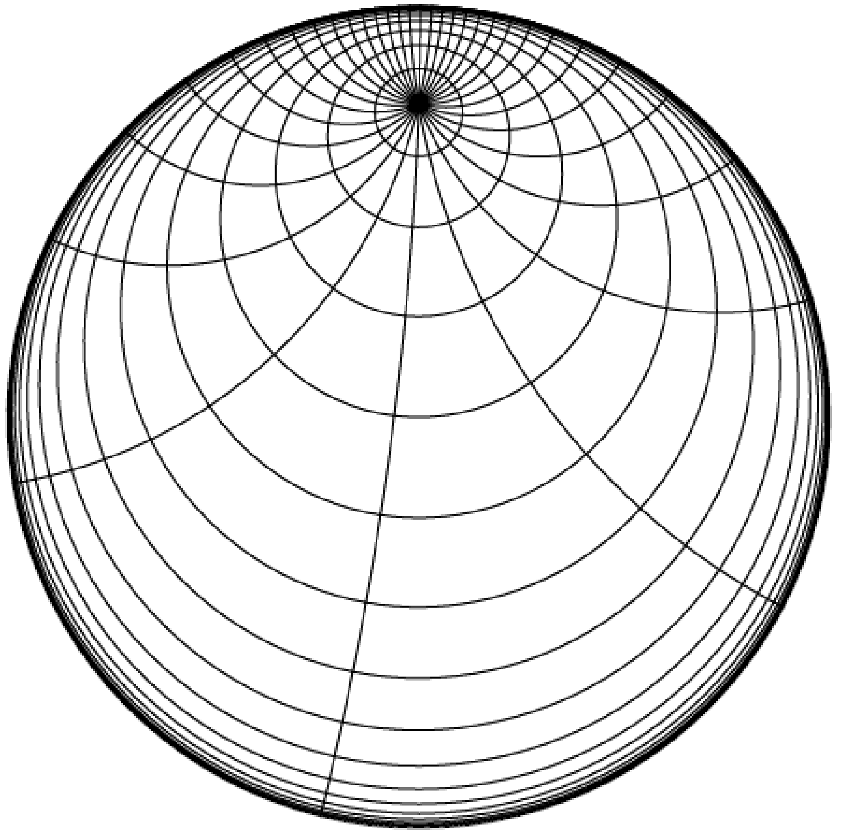
Proof will be given later in these lectures, if time permits.

THEOREM: Let $\Gamma \subset PSL(2, \mathbb{R})$ be a discrete group. The action of Γ on $\mathbb{H}^2$ **is free if and only if it does not contain elliptic elements.** If, moreover, Γ is cocompact, **all its non-trivial elements are hyperbolic.**

Proof: The first assertion is clear, because **elliptic elements have fixed points on $\mathbb{H}^2$, hyperbolic and parabolic act without fixed points.**

To prove the second, let $\gamma \in \Gamma = \pi_1(S)$. Then corresponding class in $\pi_1(S)$ can be represented by a closed geodesic $s \subset S$ **(prove it).** Let $\tilde{s} \subset \mathbb{H}^2$ be its preimage. Since $\tilde{s}$ contains x and $\gamma(x)$, **the action of γ preserves the geodesic $\tilde{s}$,** hence γ is hyperbolic. ■

Elliptic isometry



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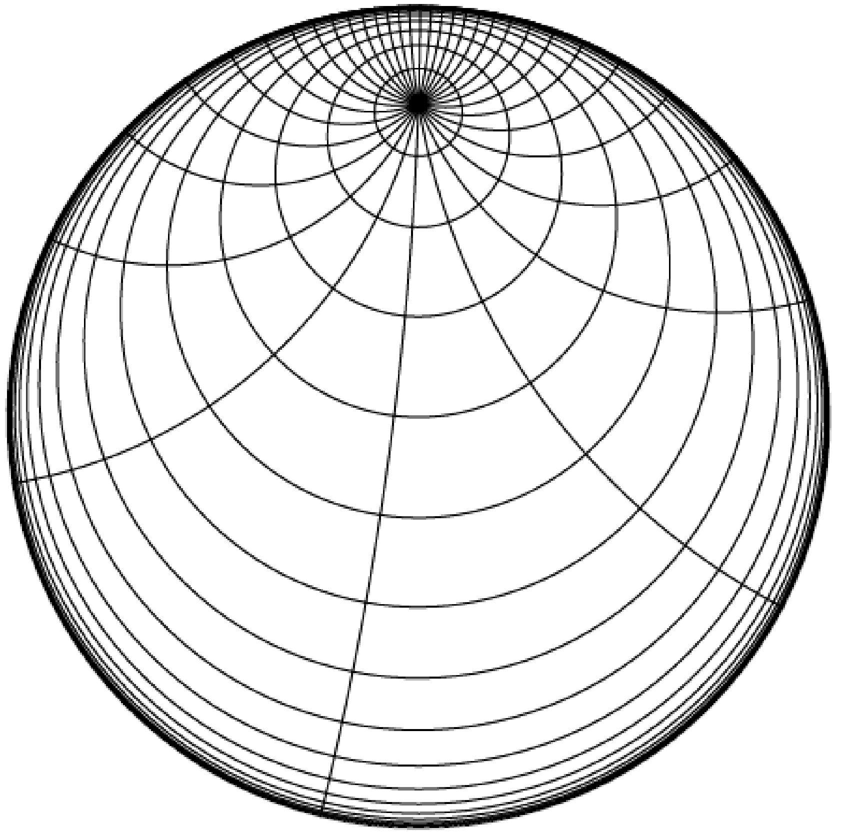

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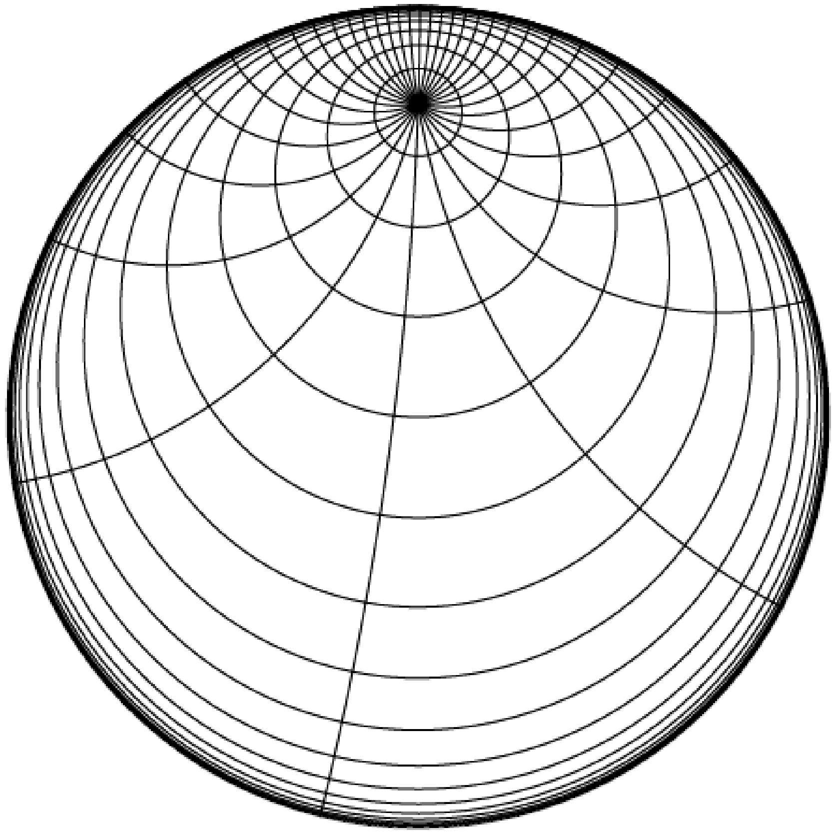

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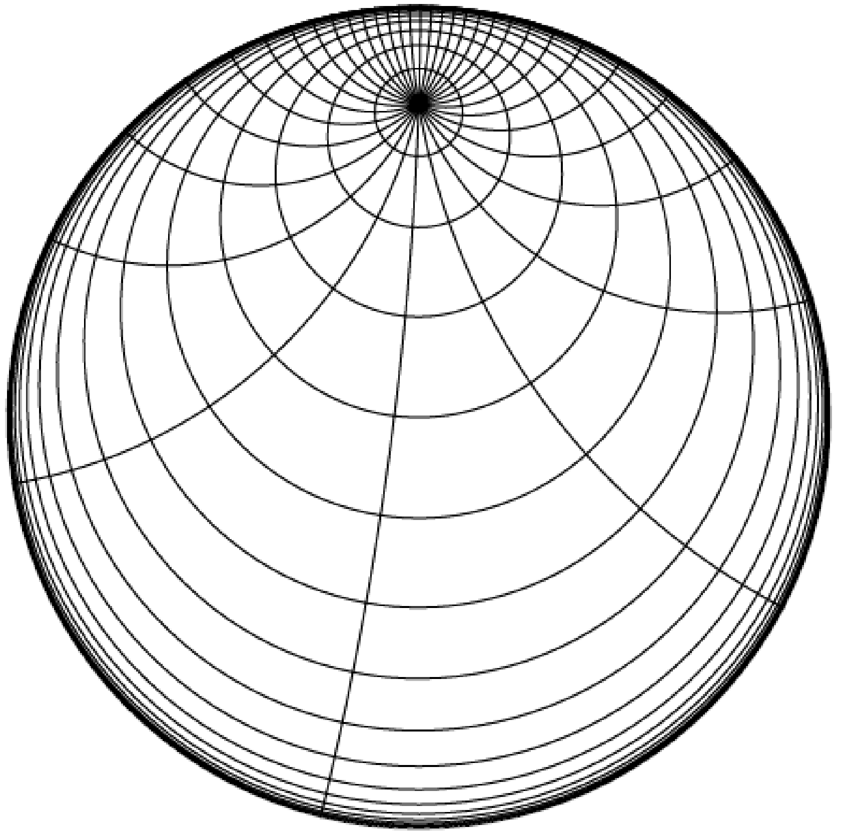

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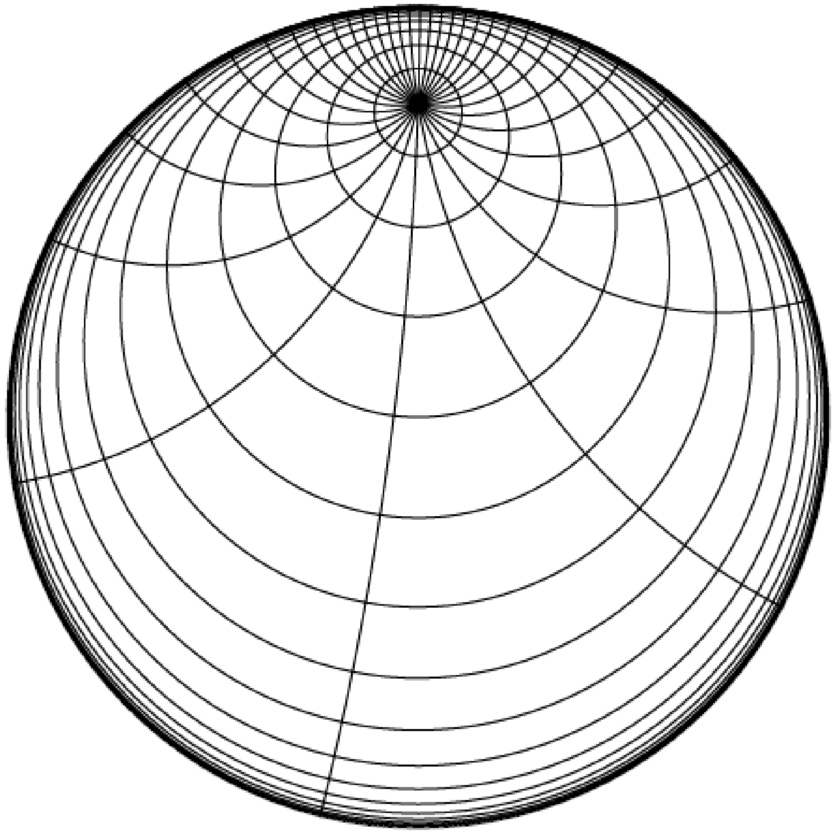

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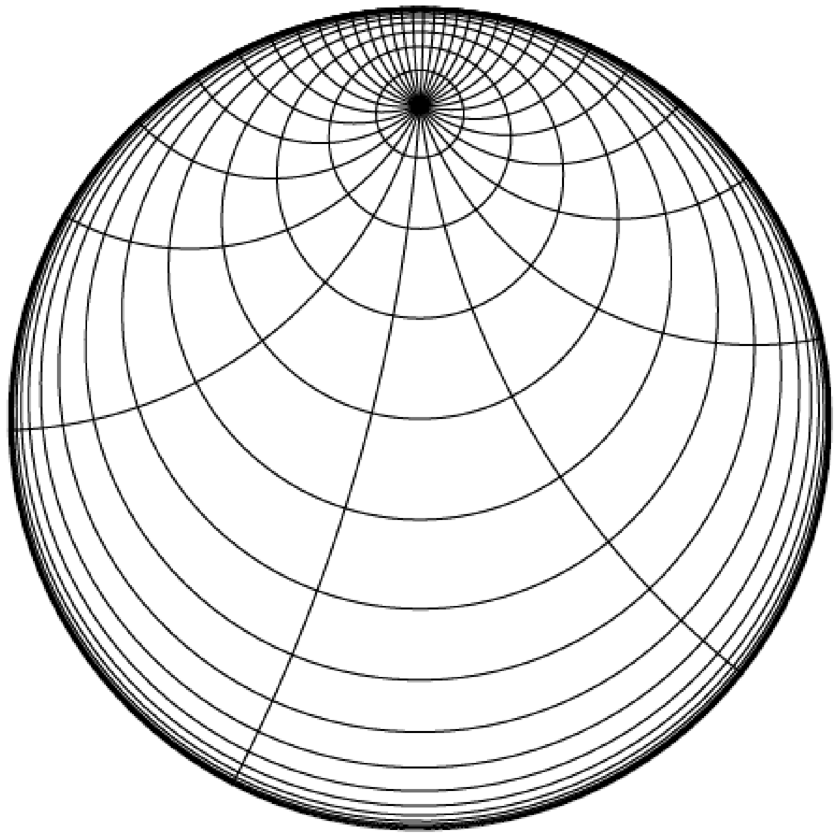

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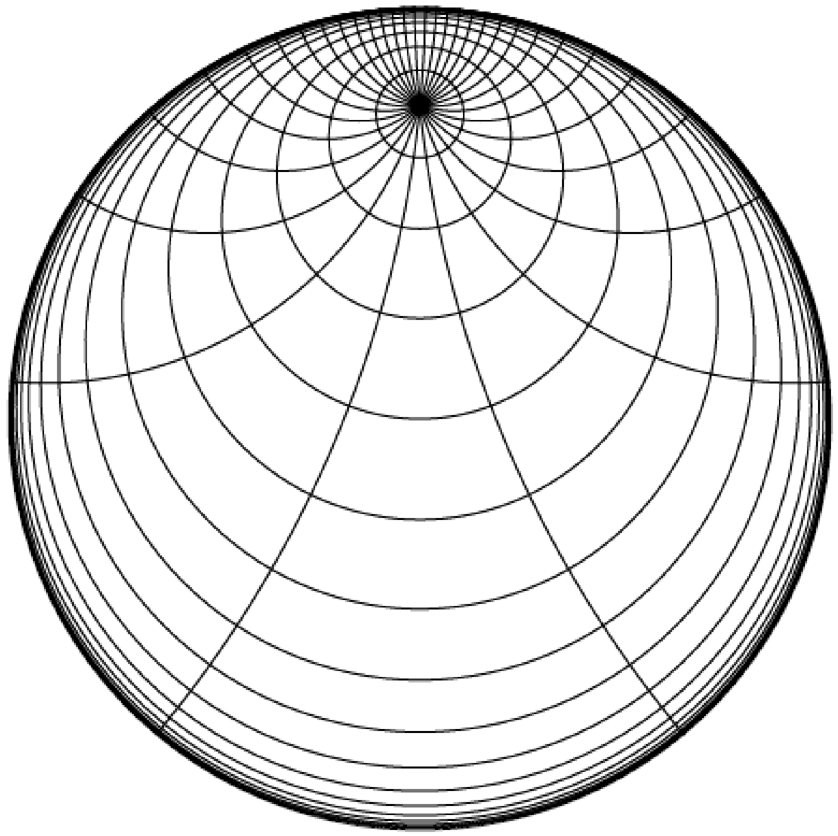

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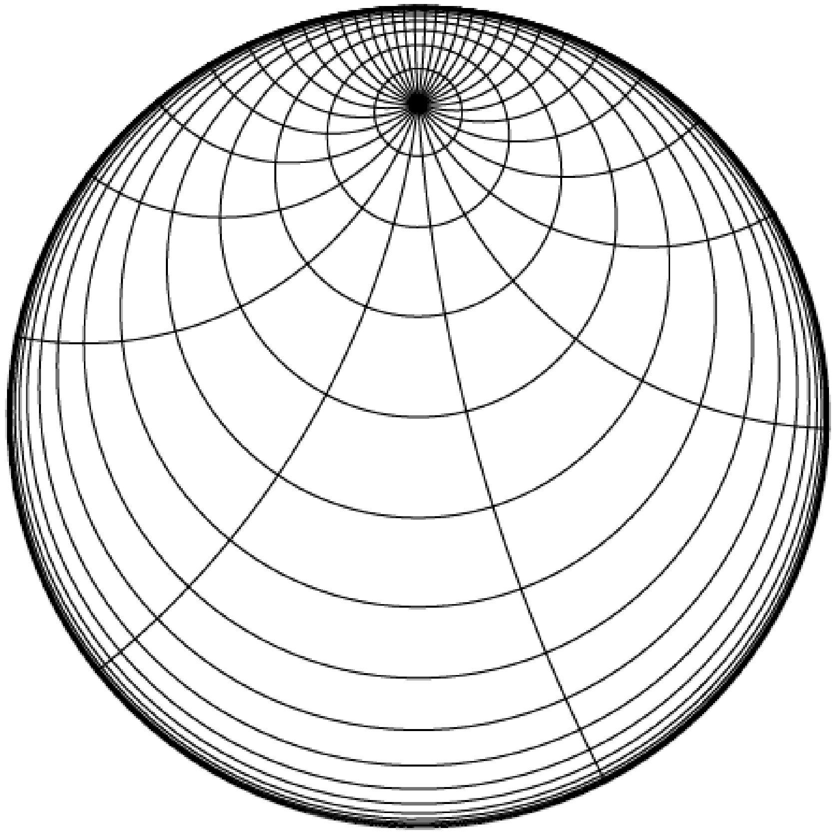

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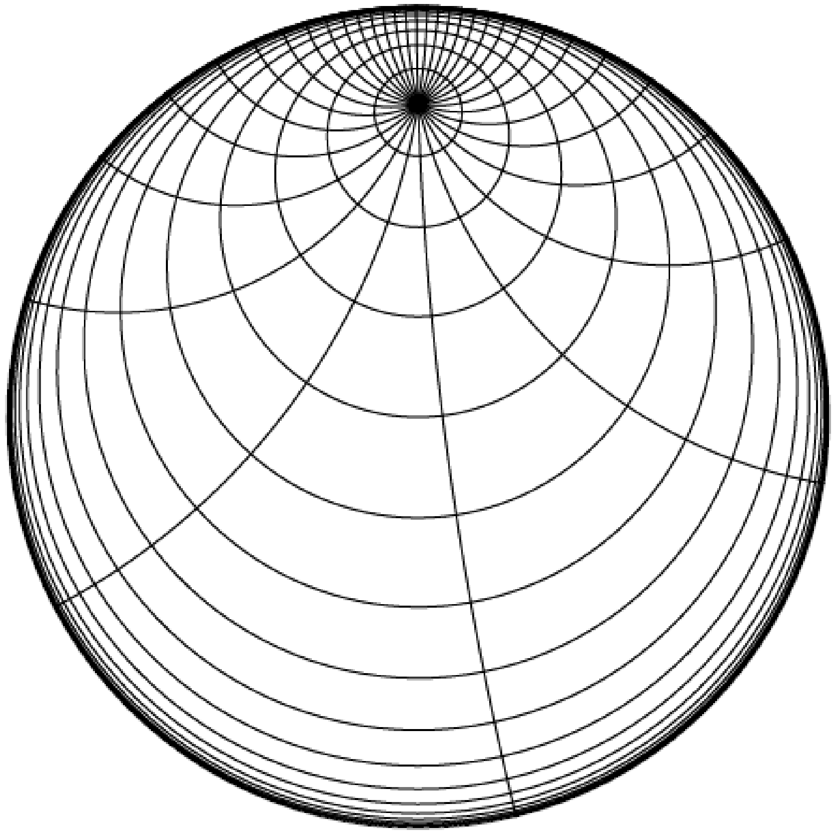

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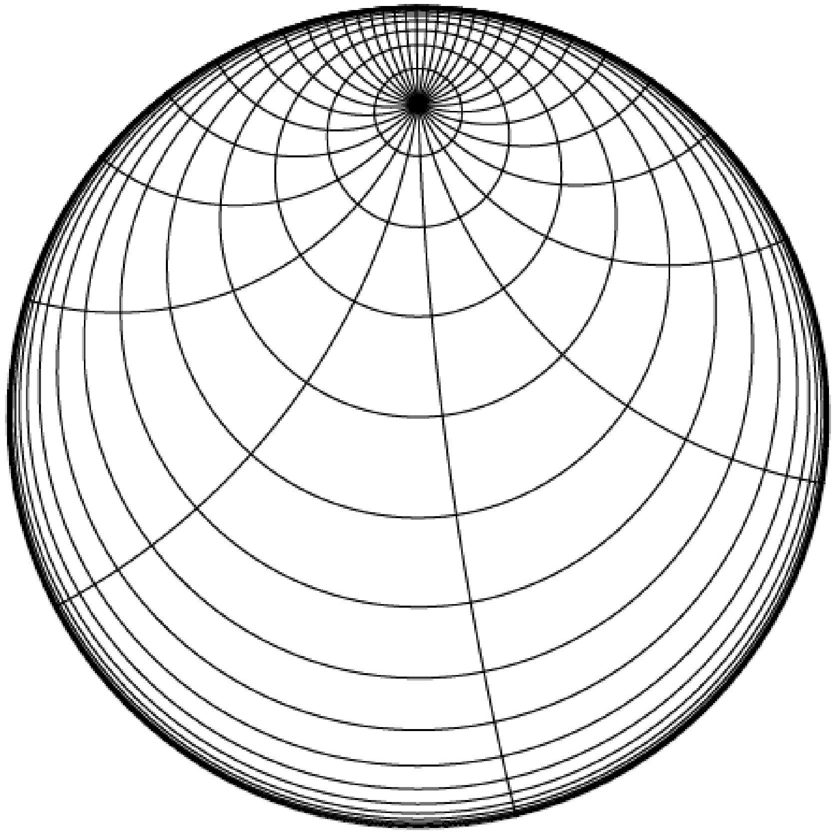

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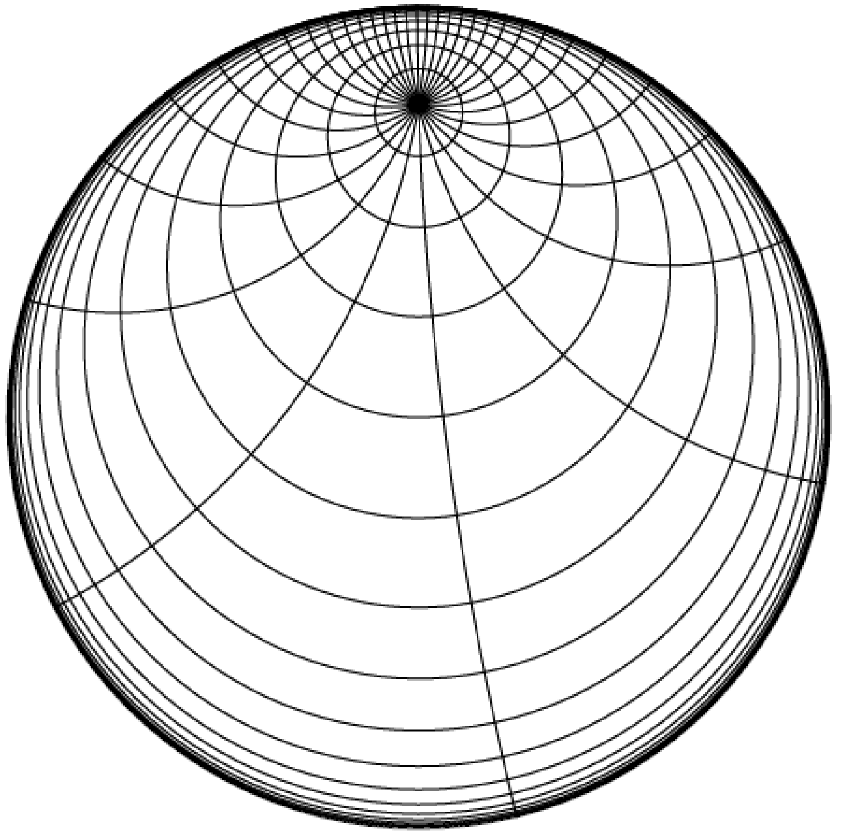

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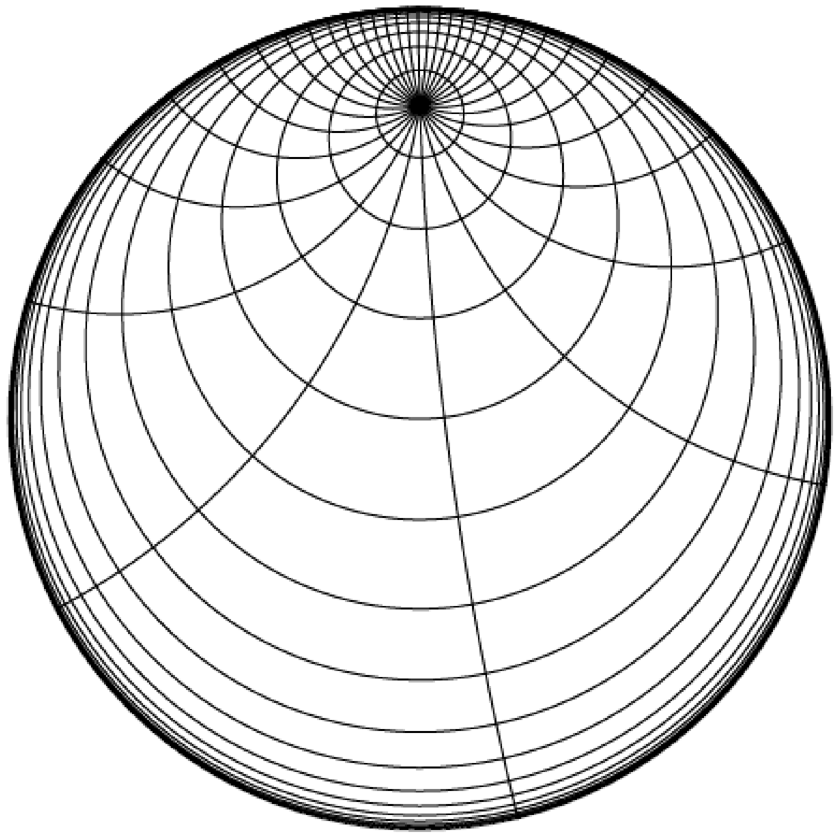

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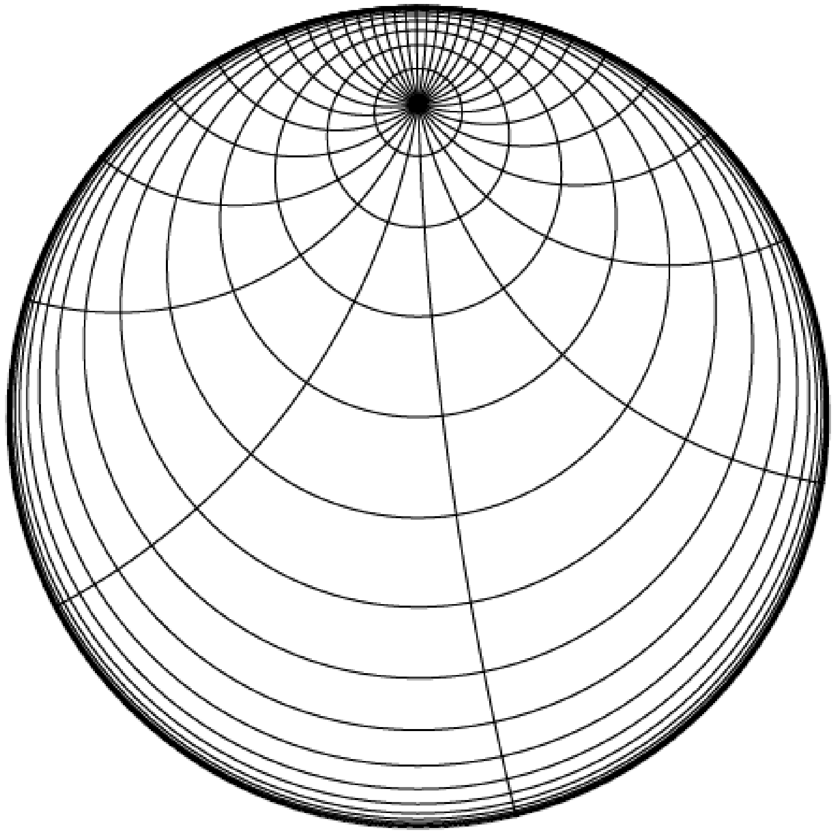
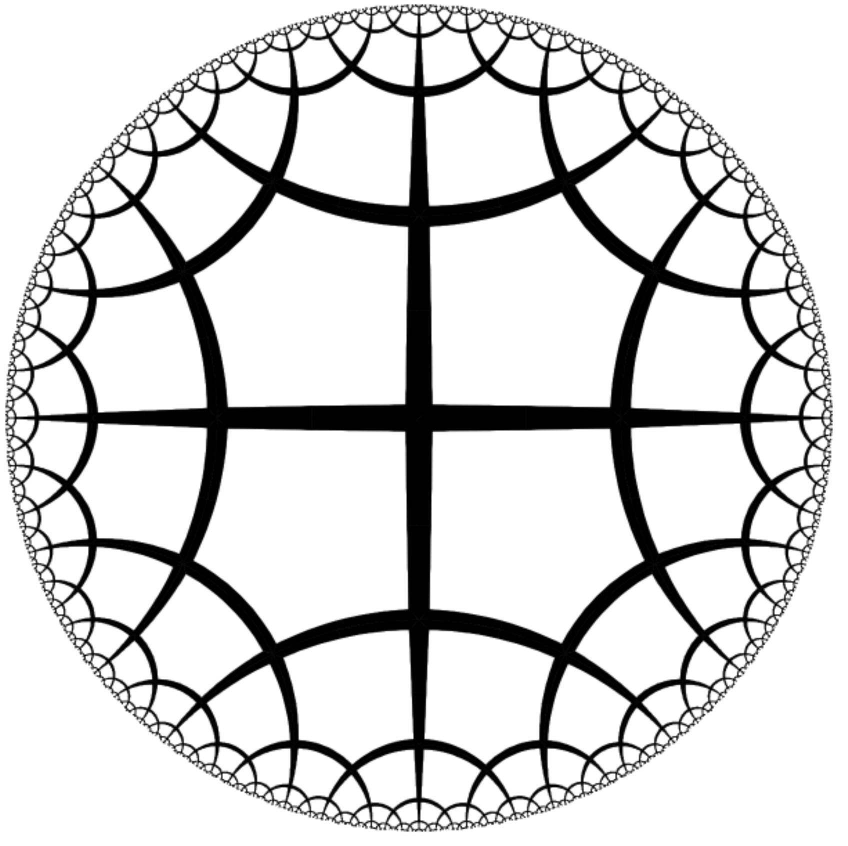


Parabolic isometry



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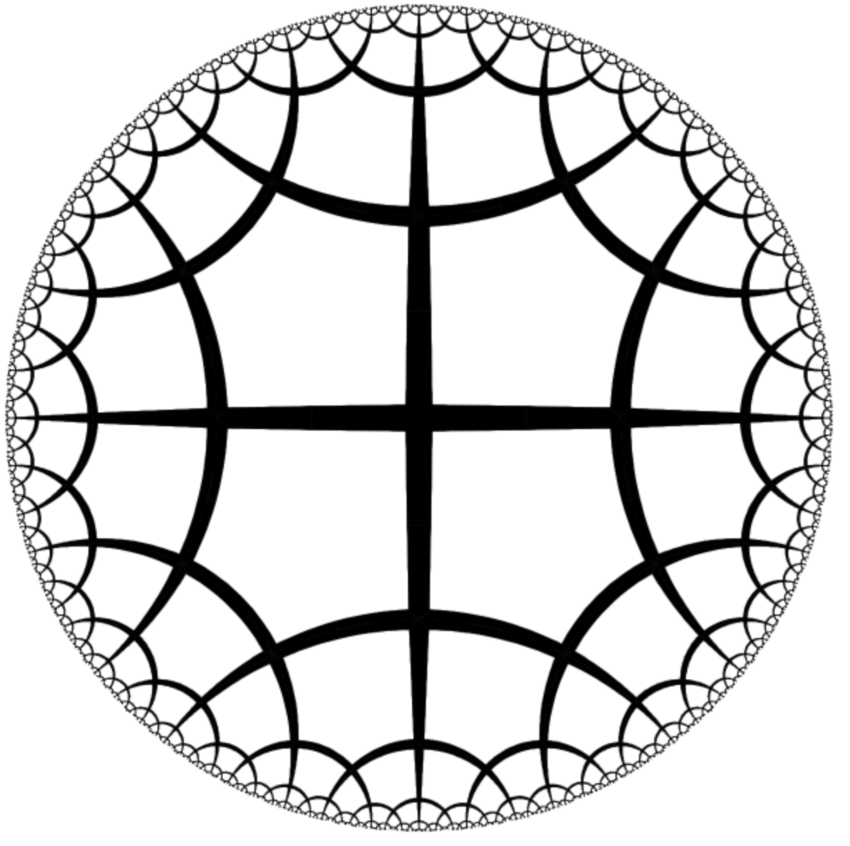

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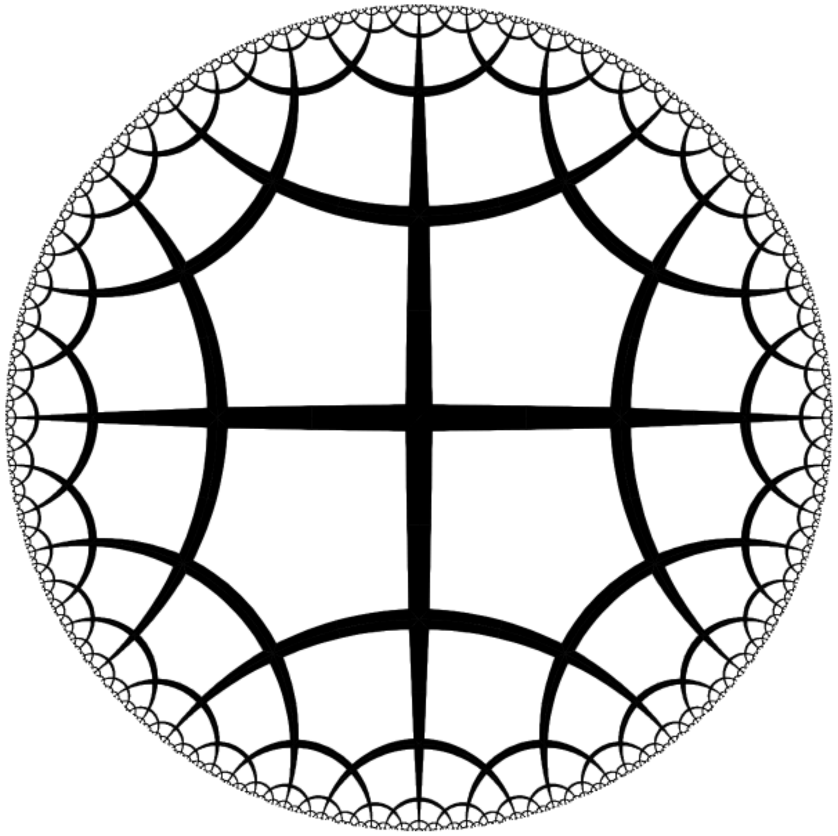

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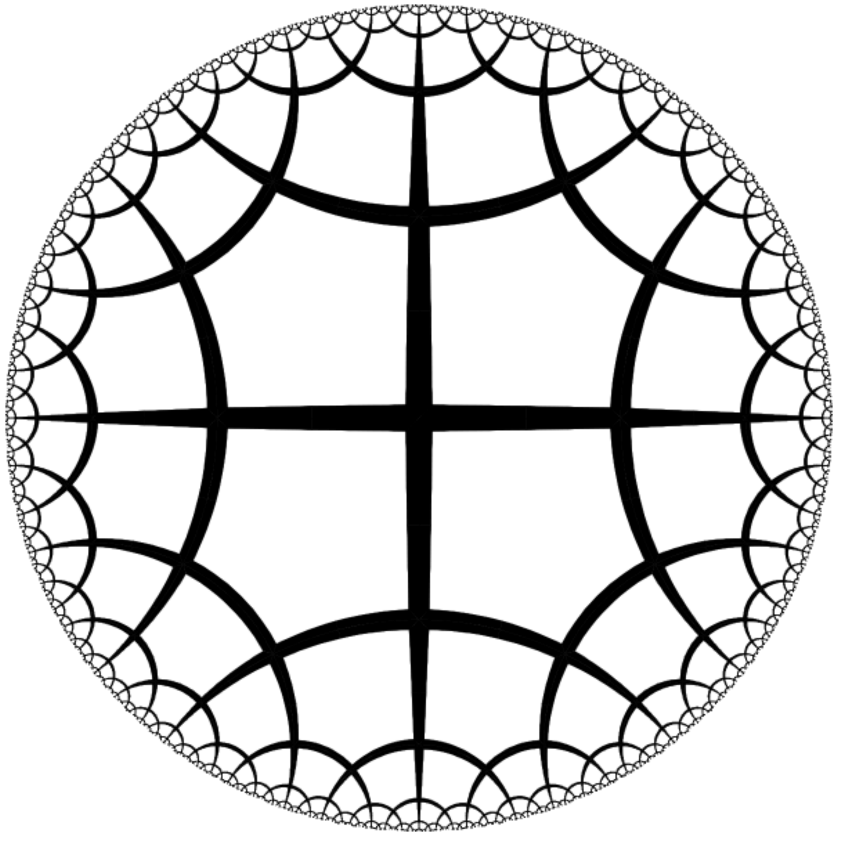

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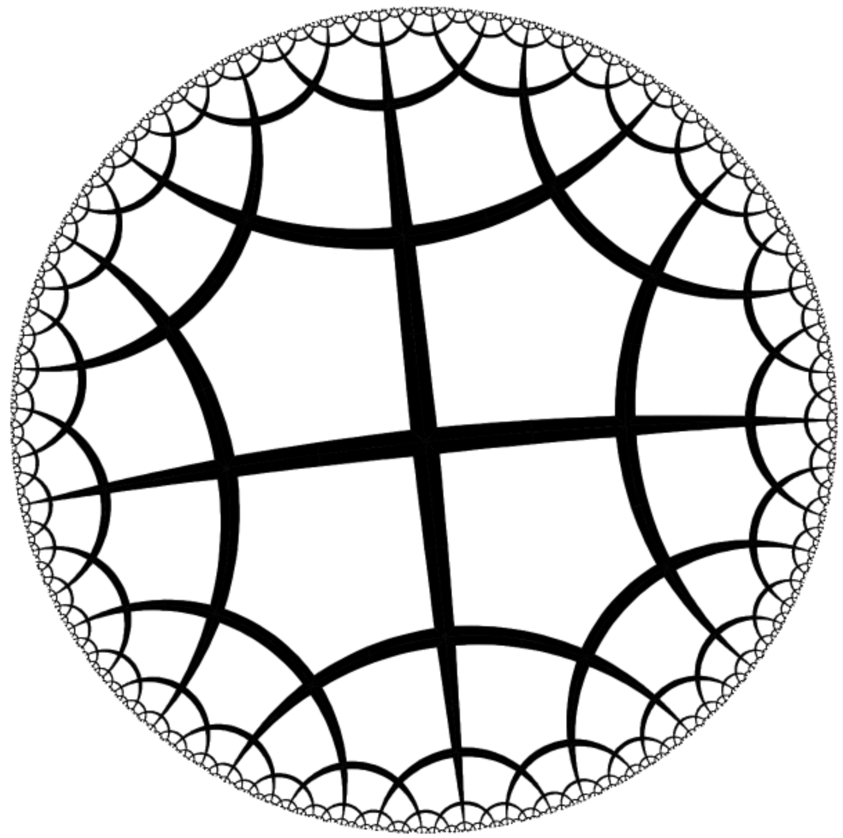

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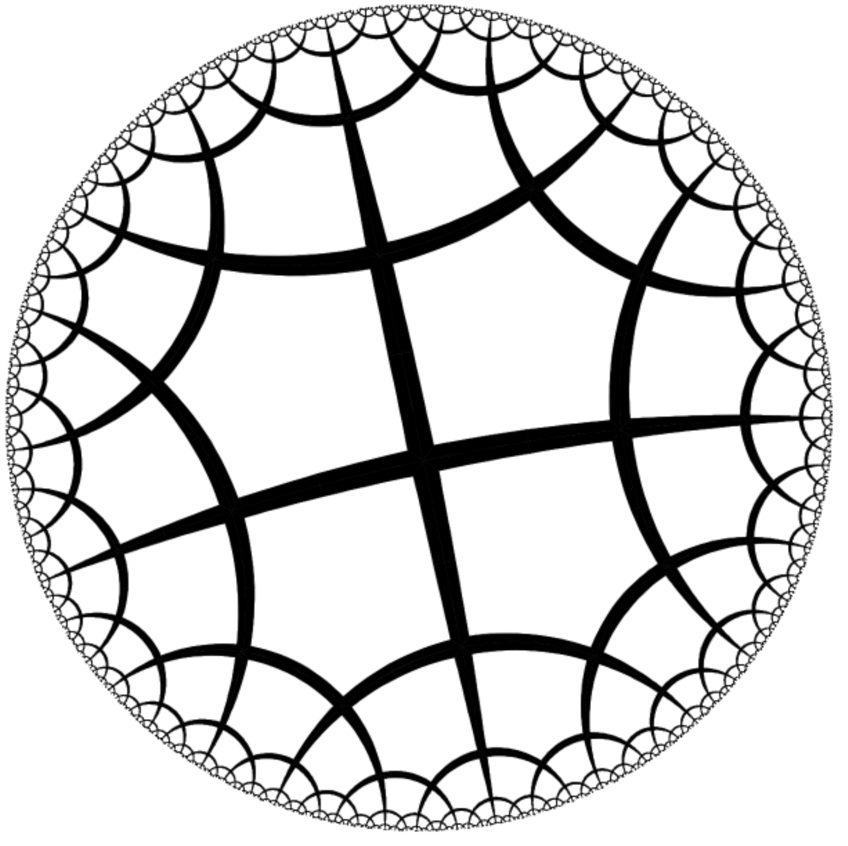

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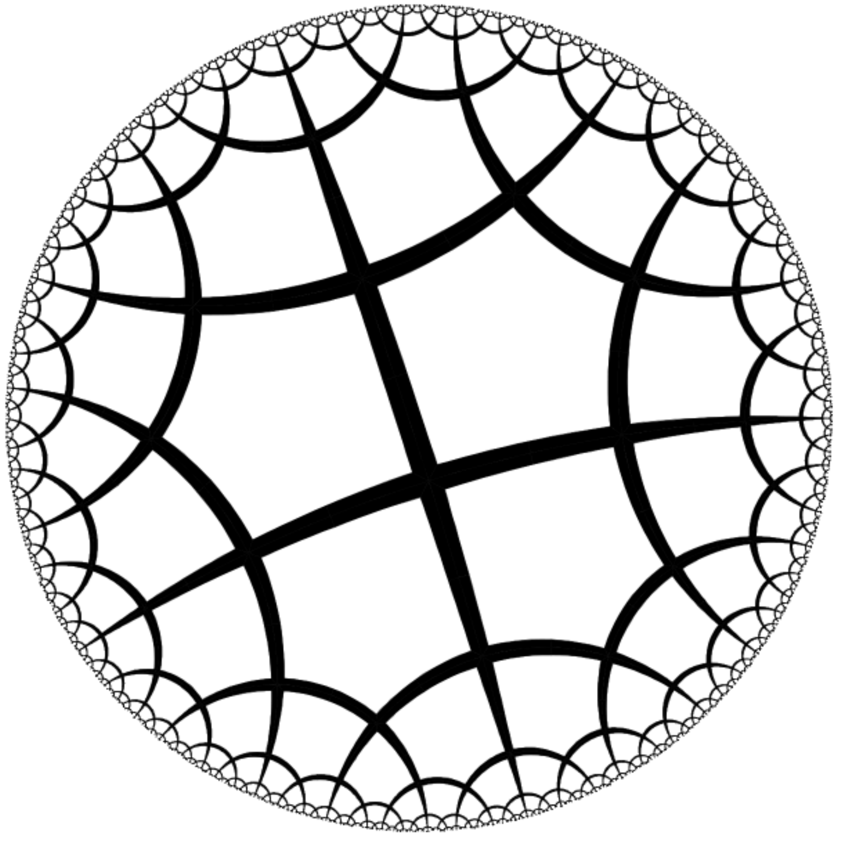

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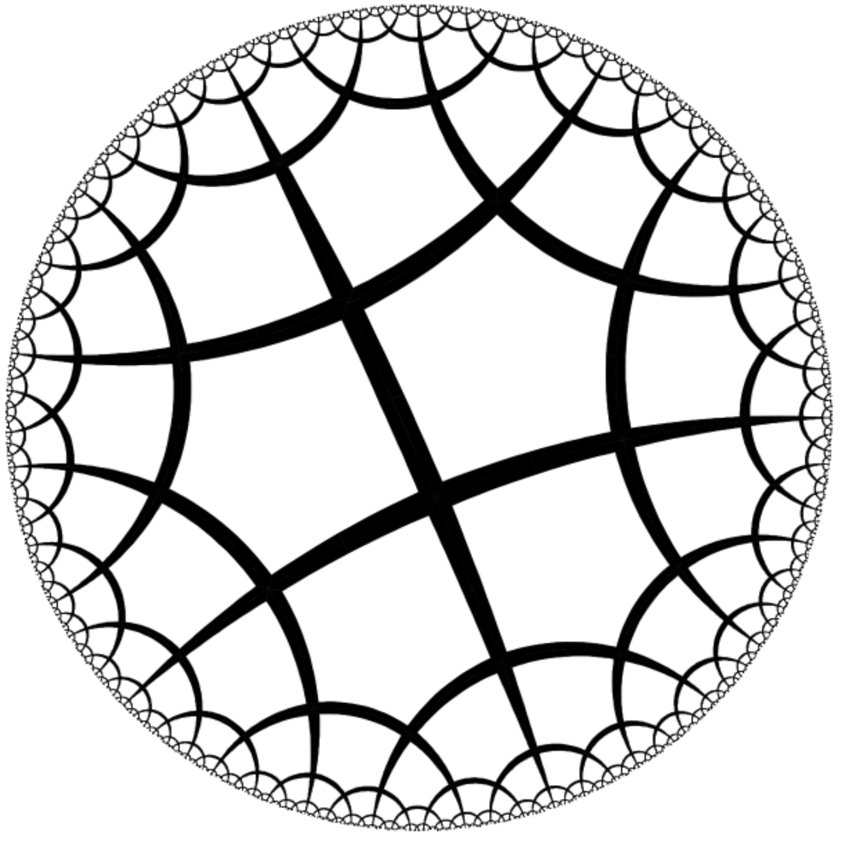

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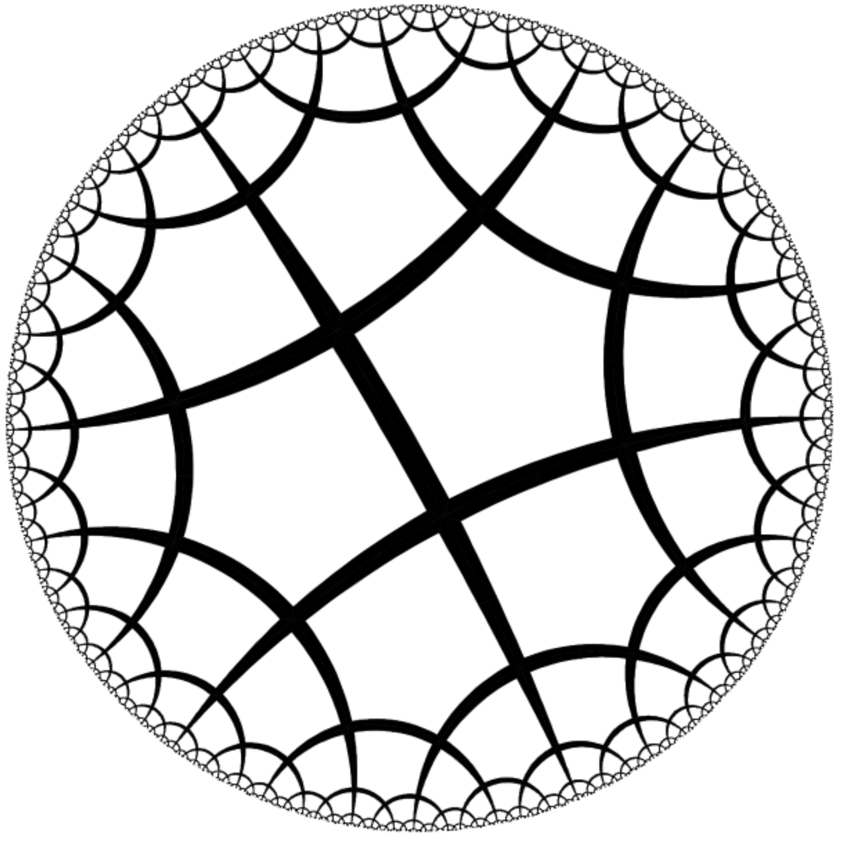

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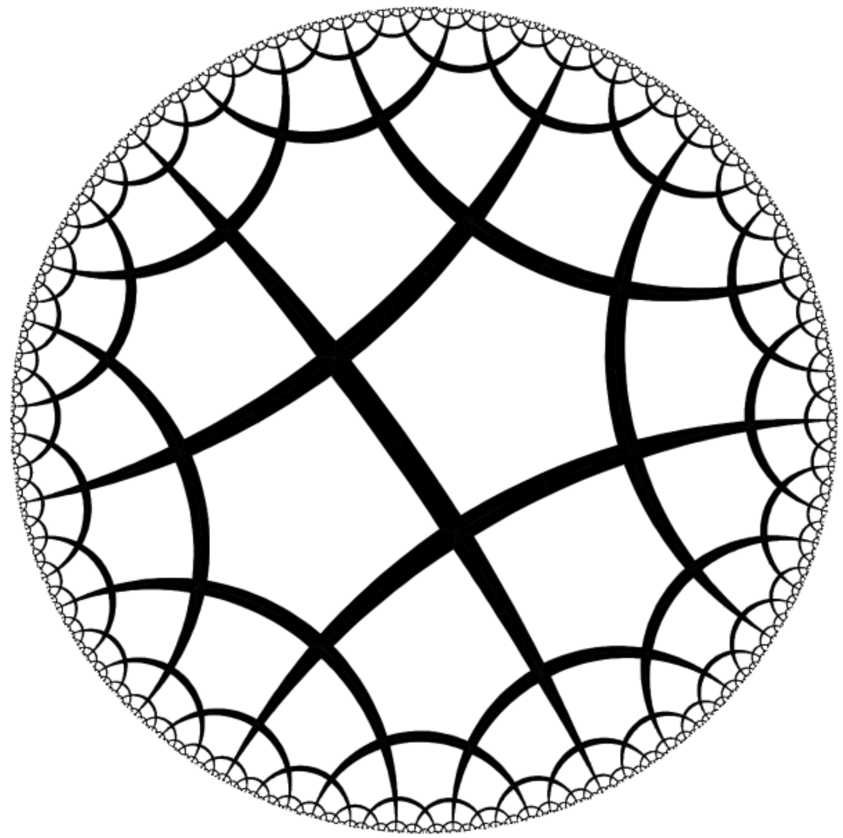

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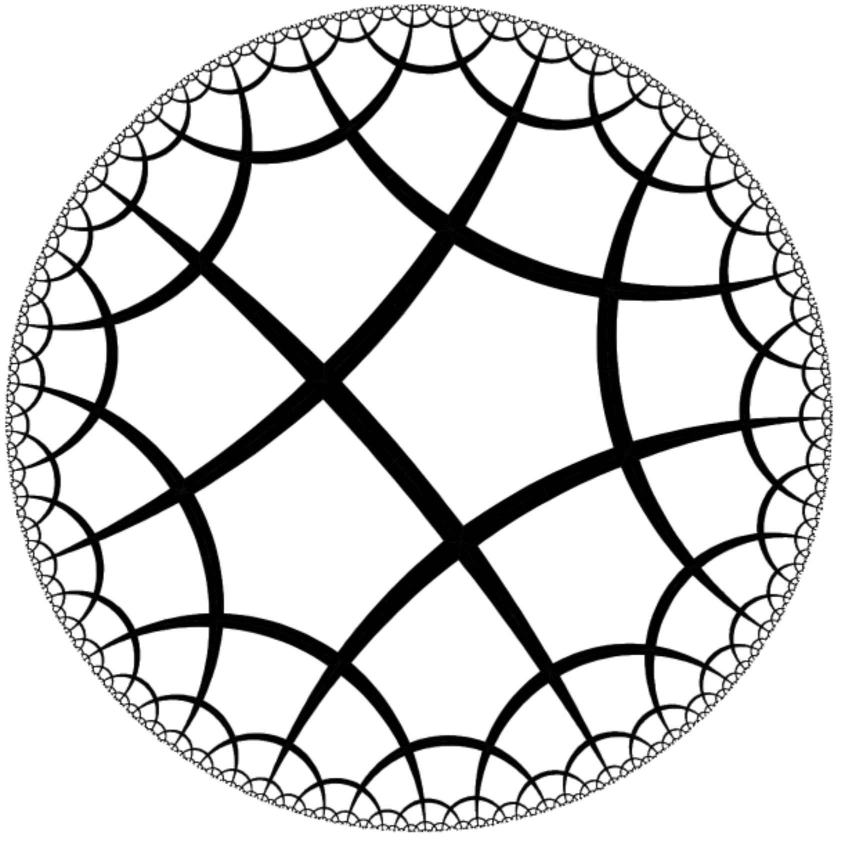

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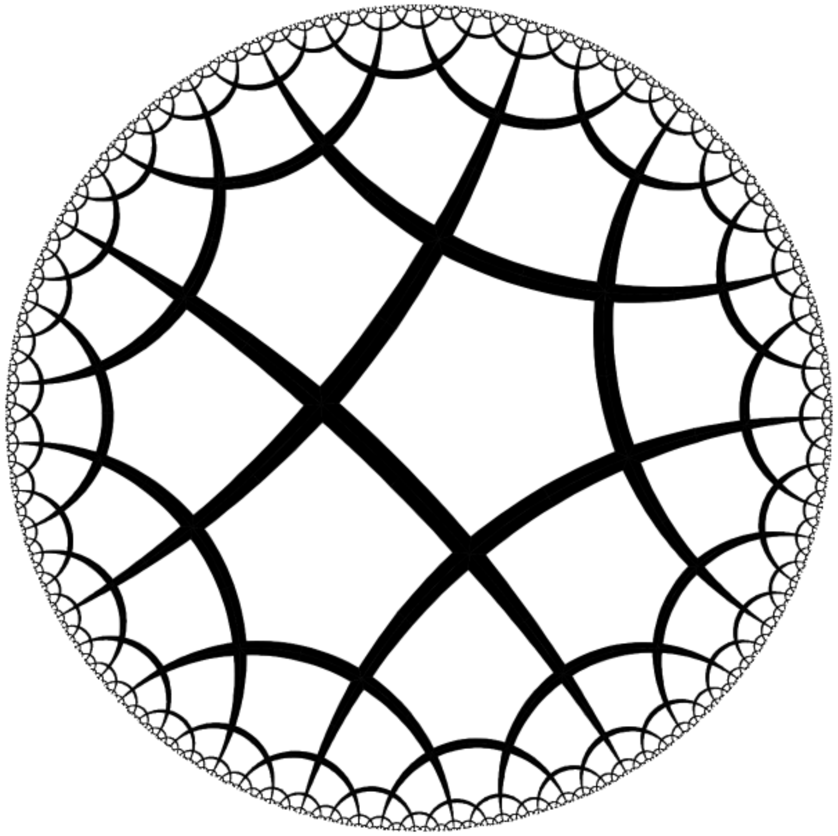

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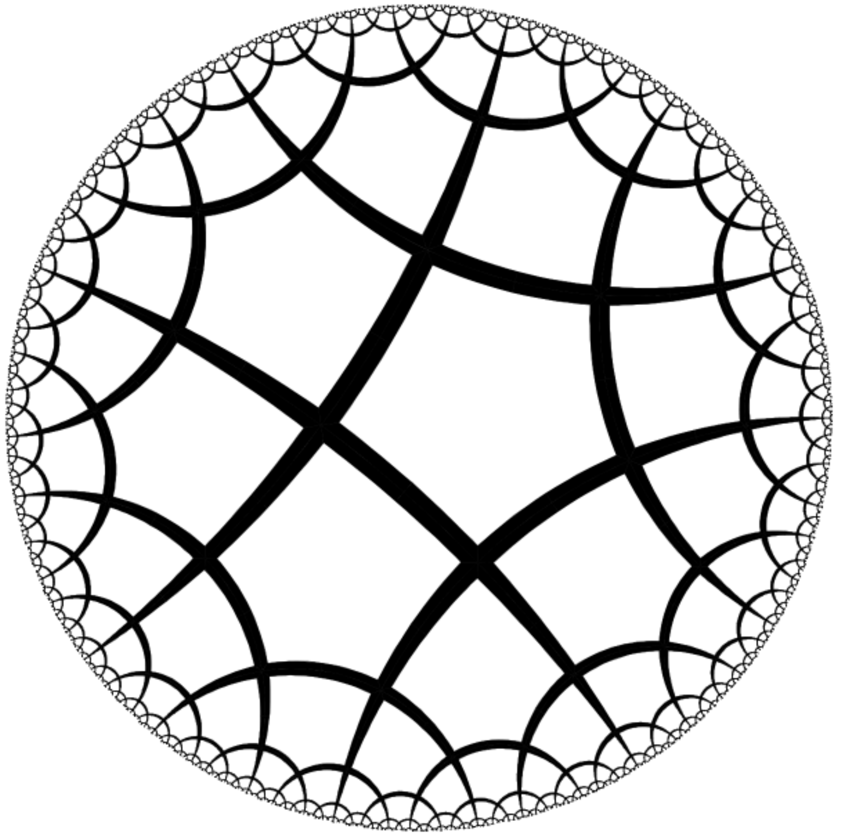

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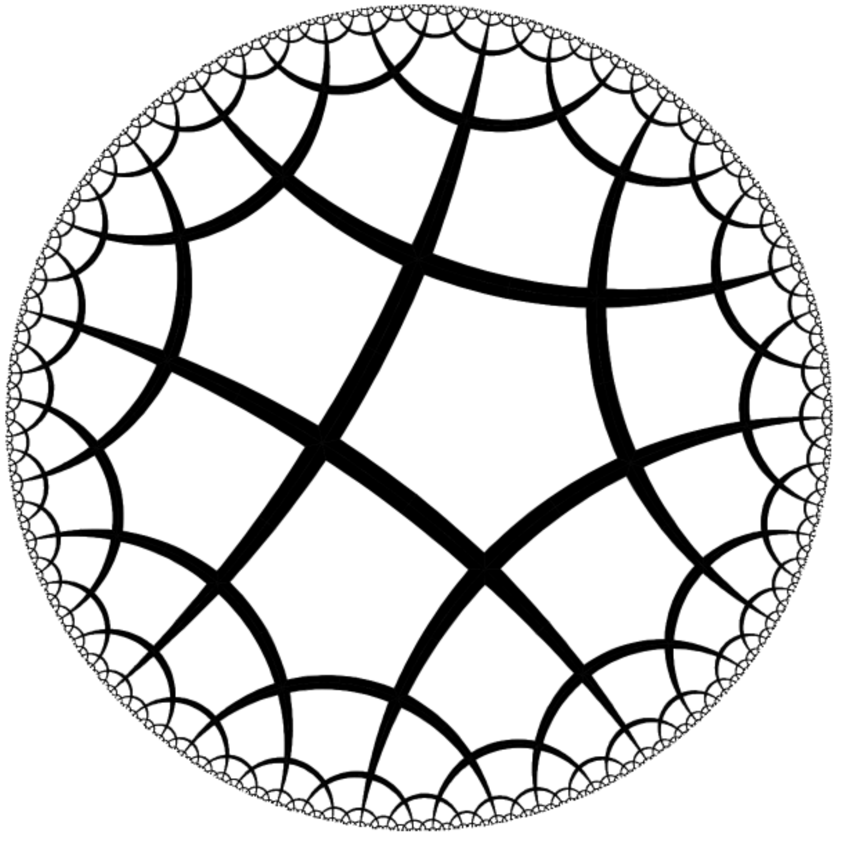

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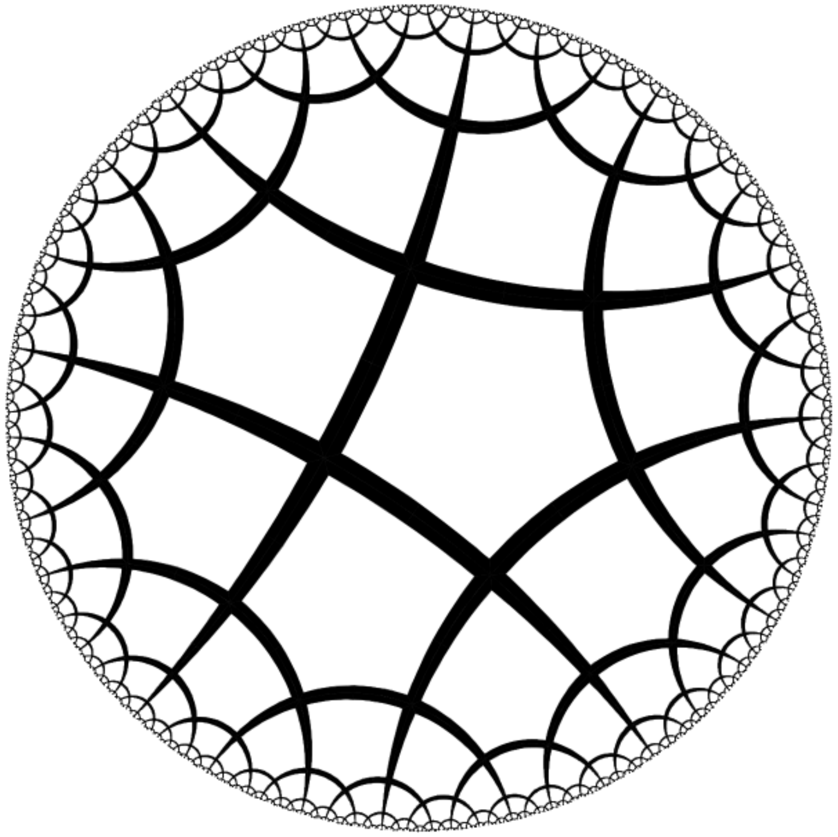

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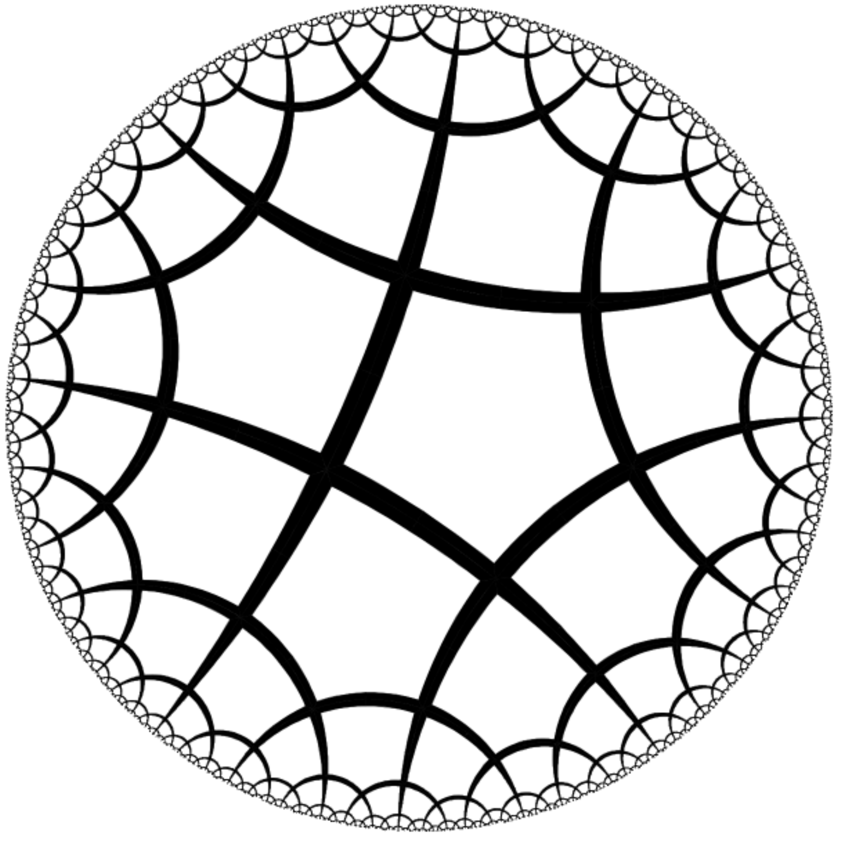
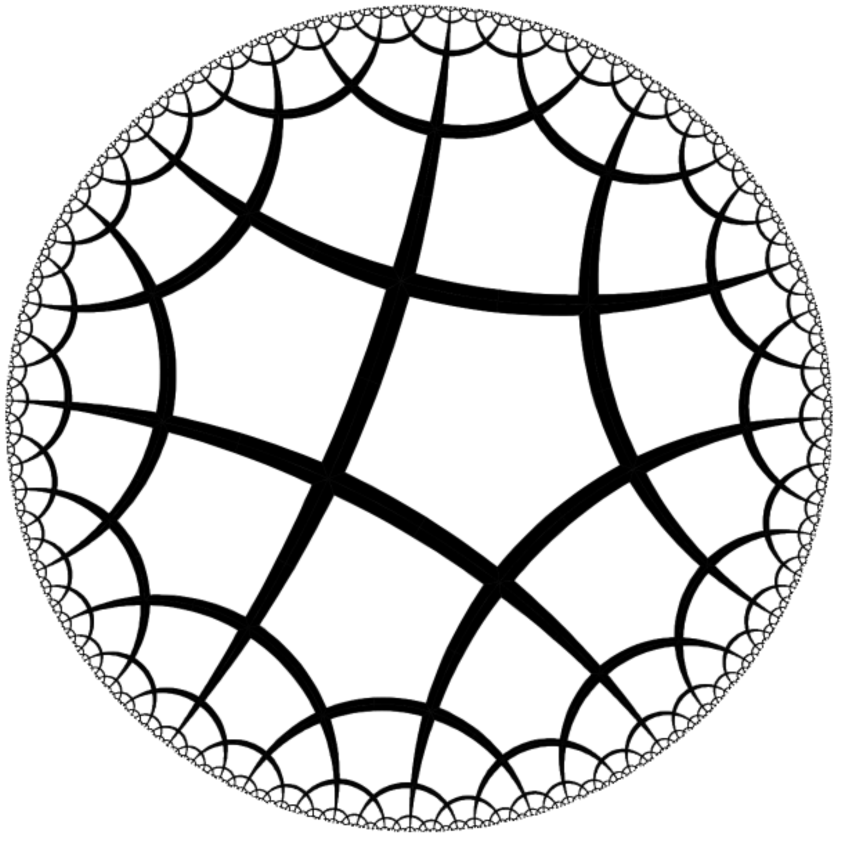


Parabolic isometry



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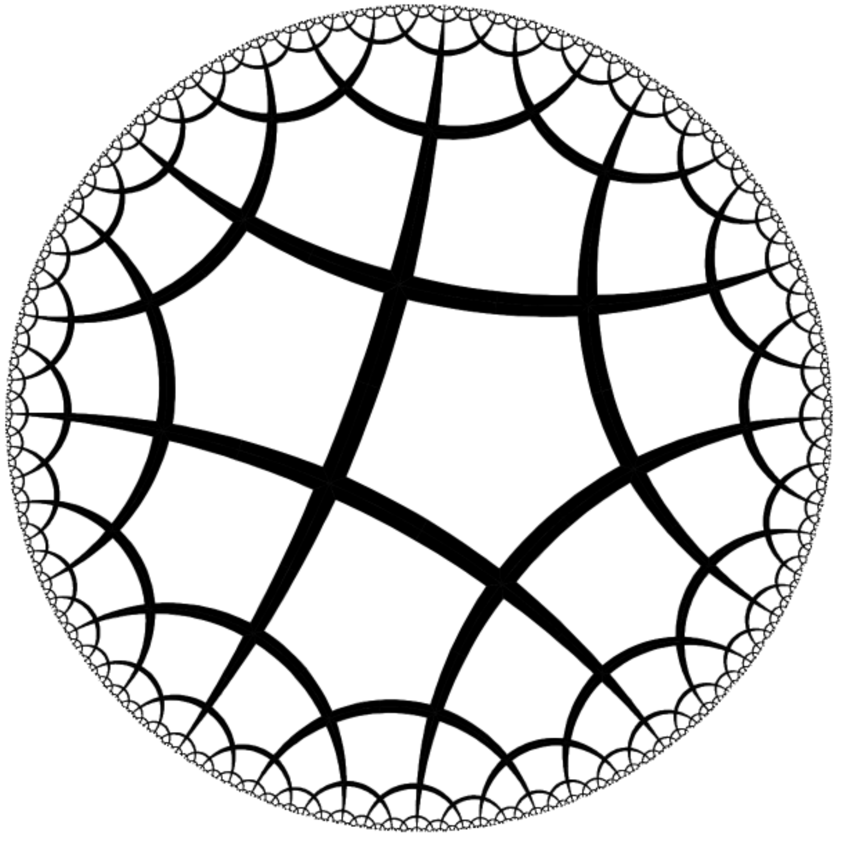

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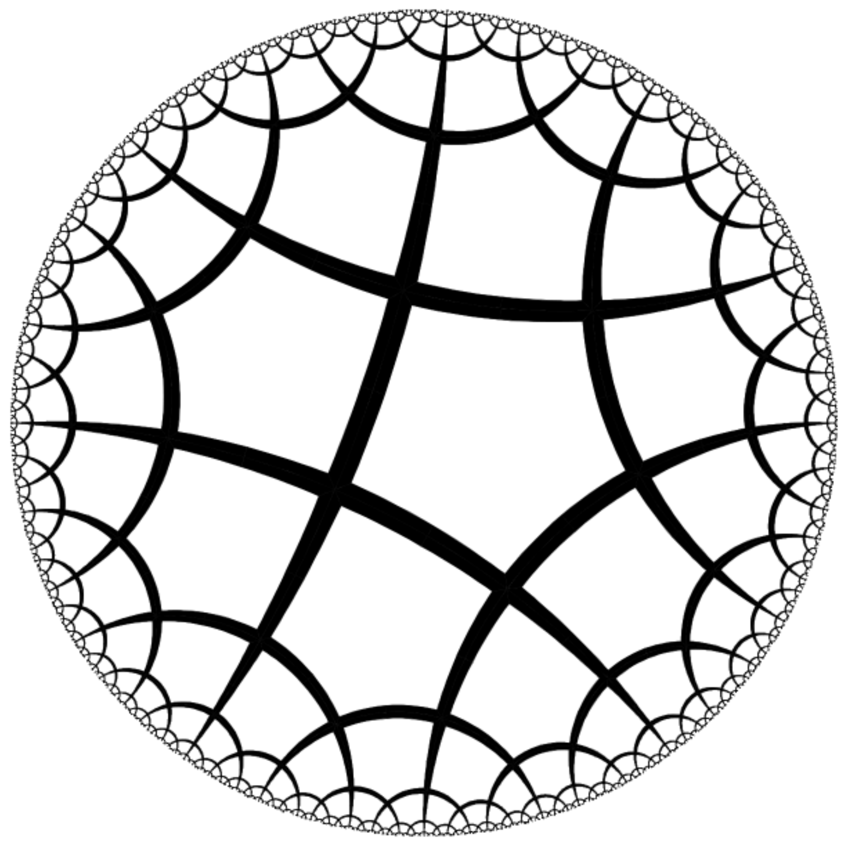

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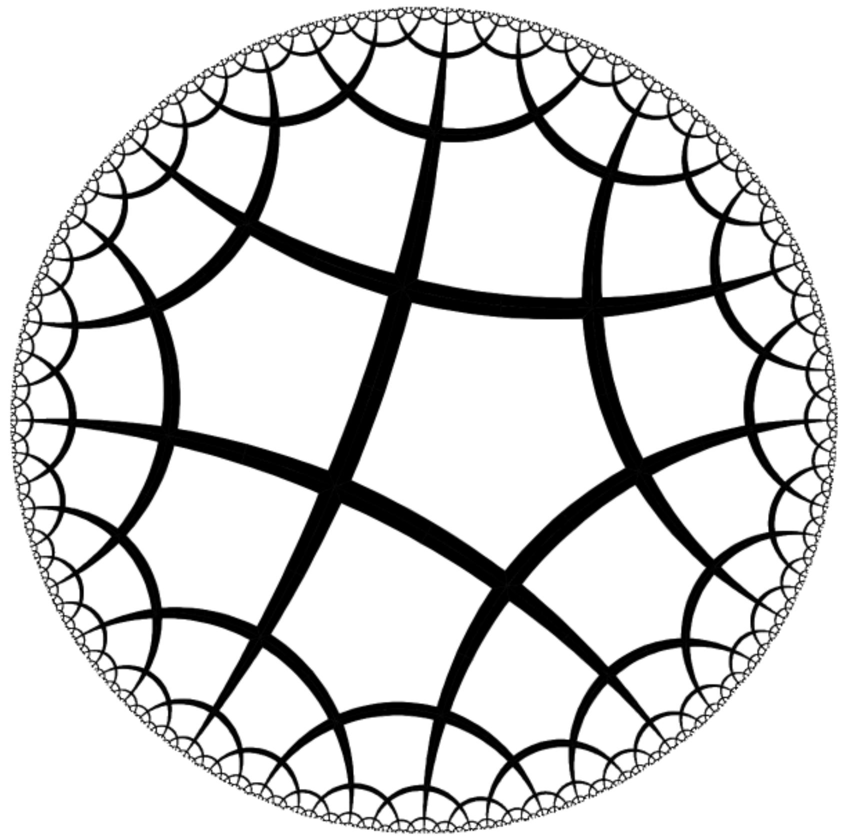

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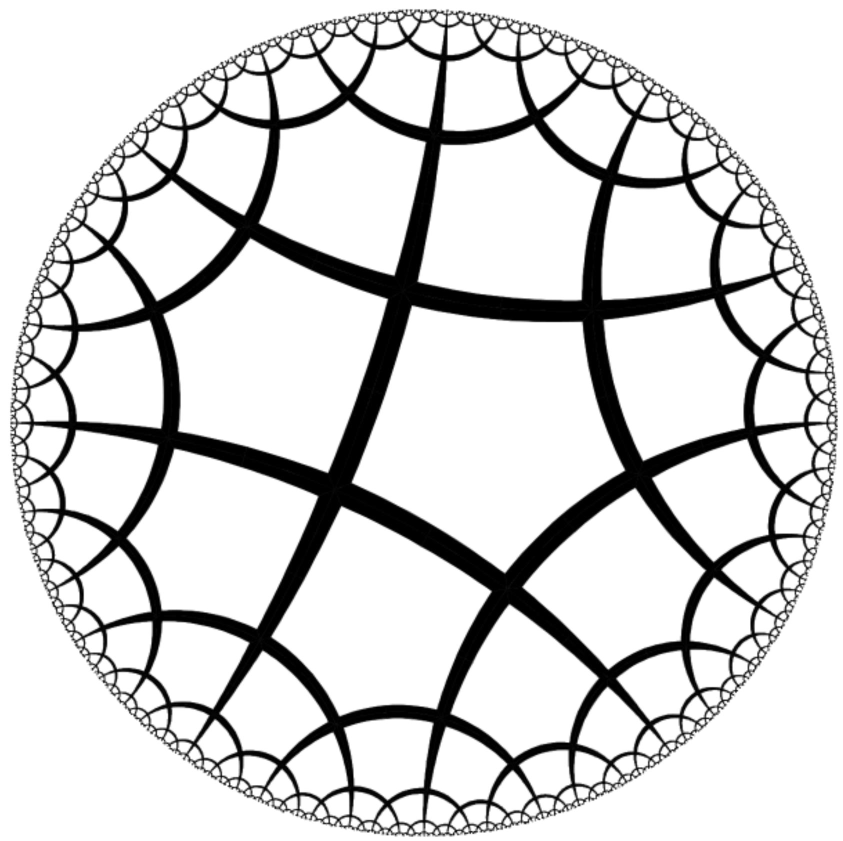

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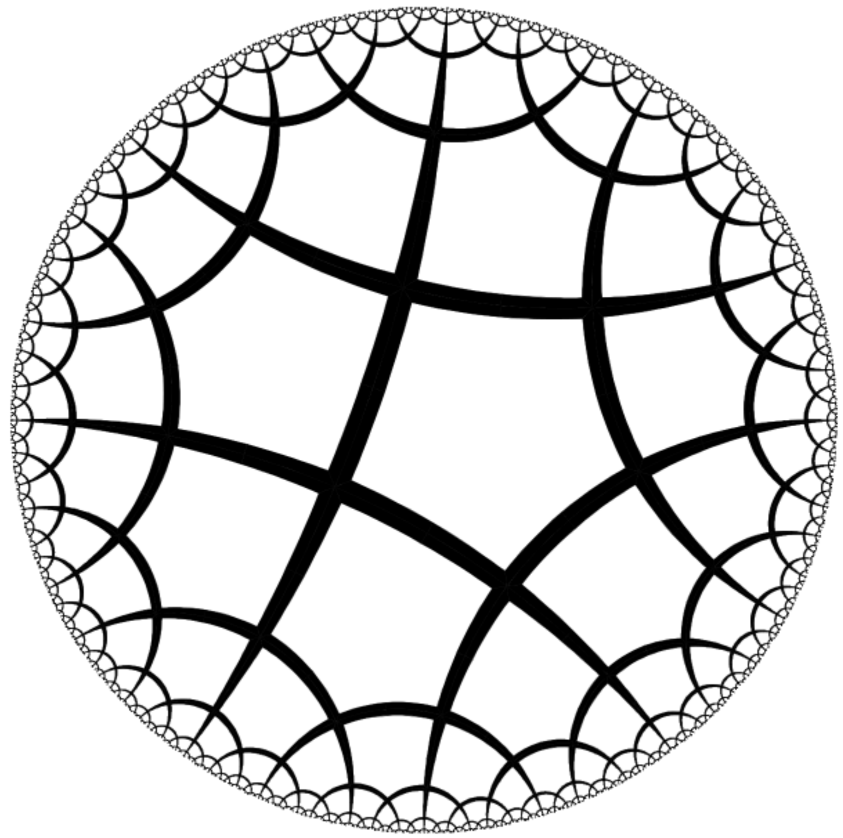

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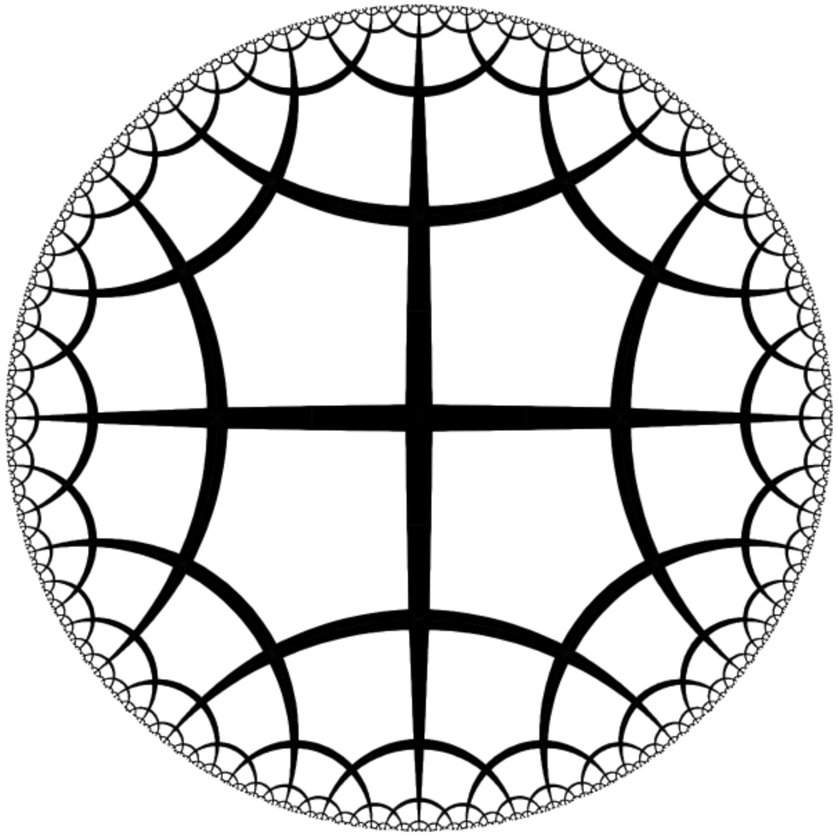

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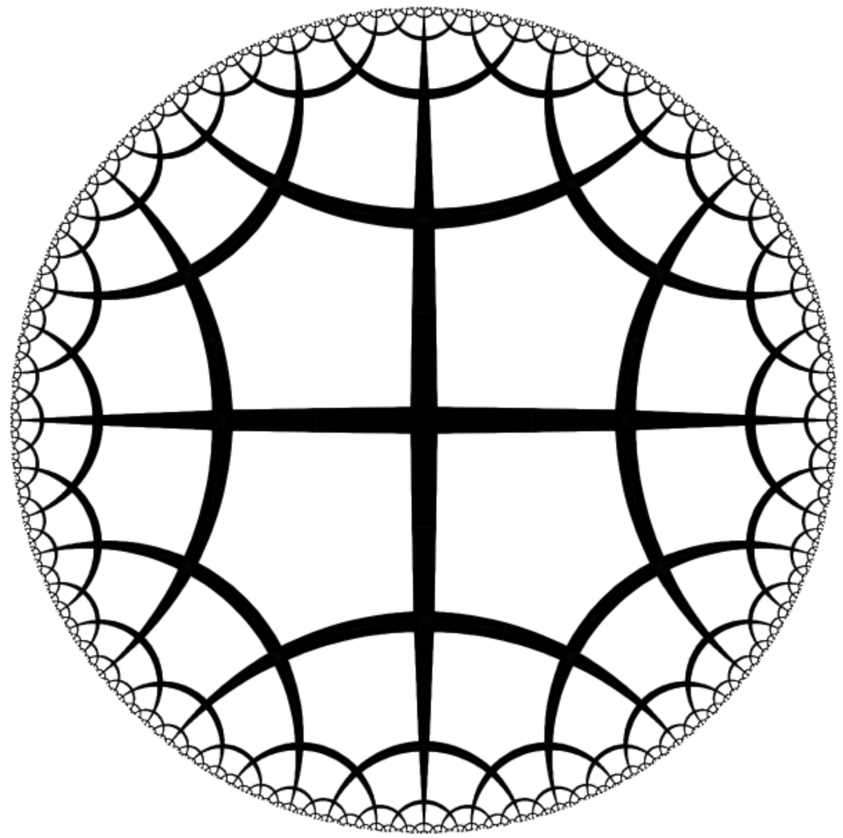

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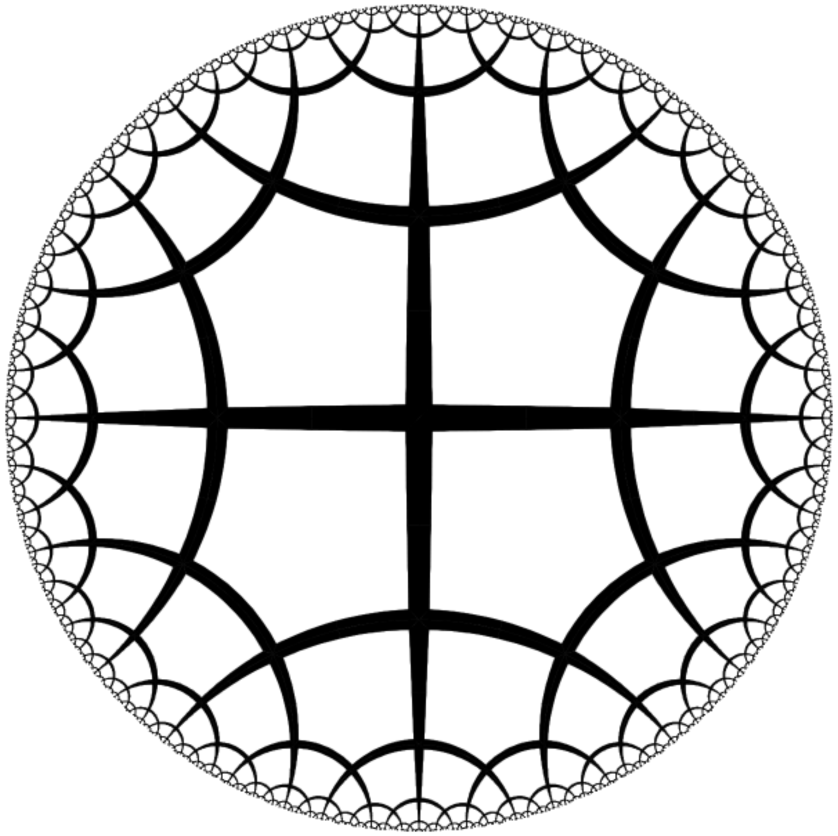

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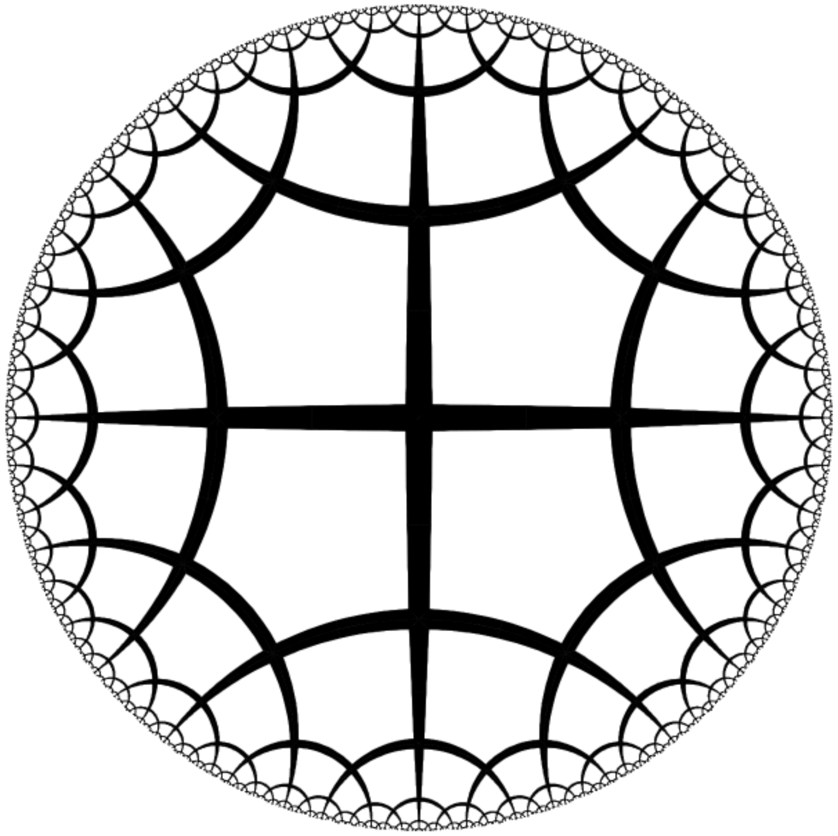

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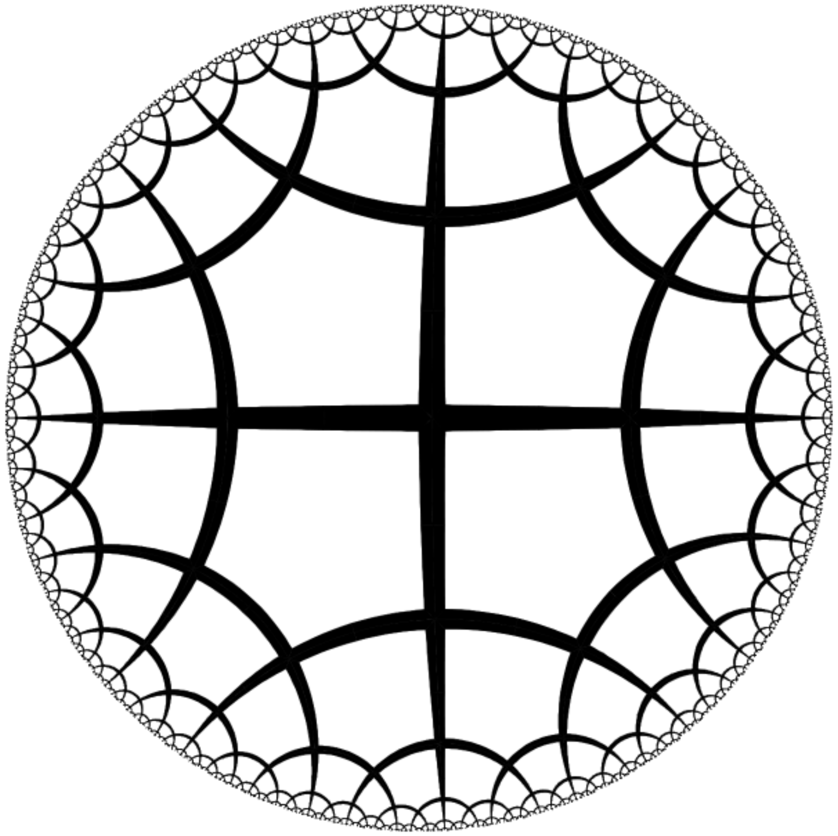

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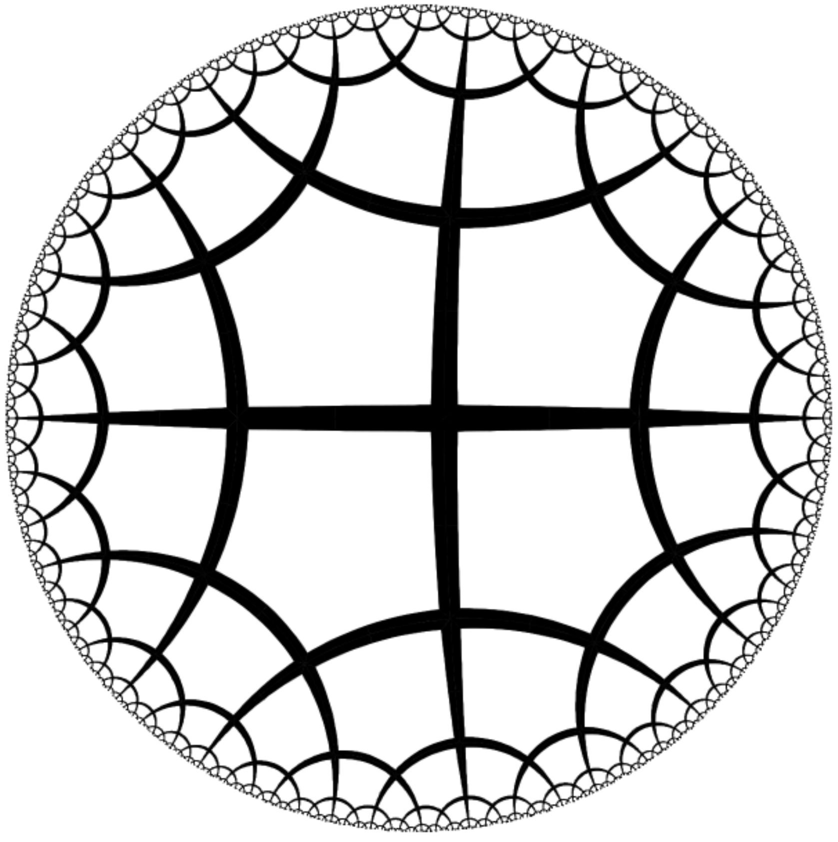

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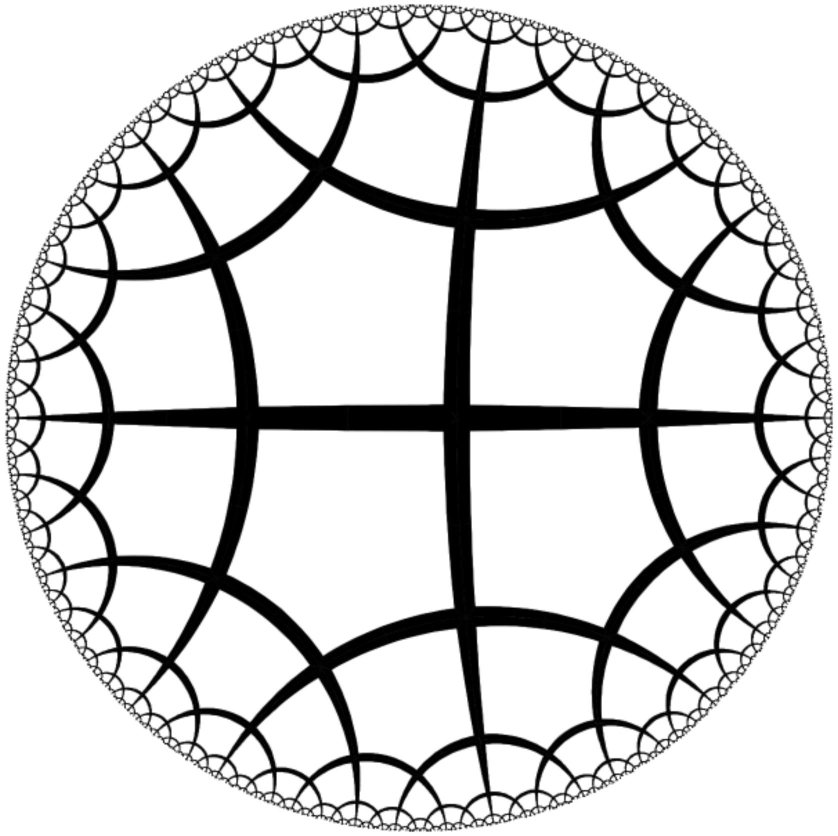

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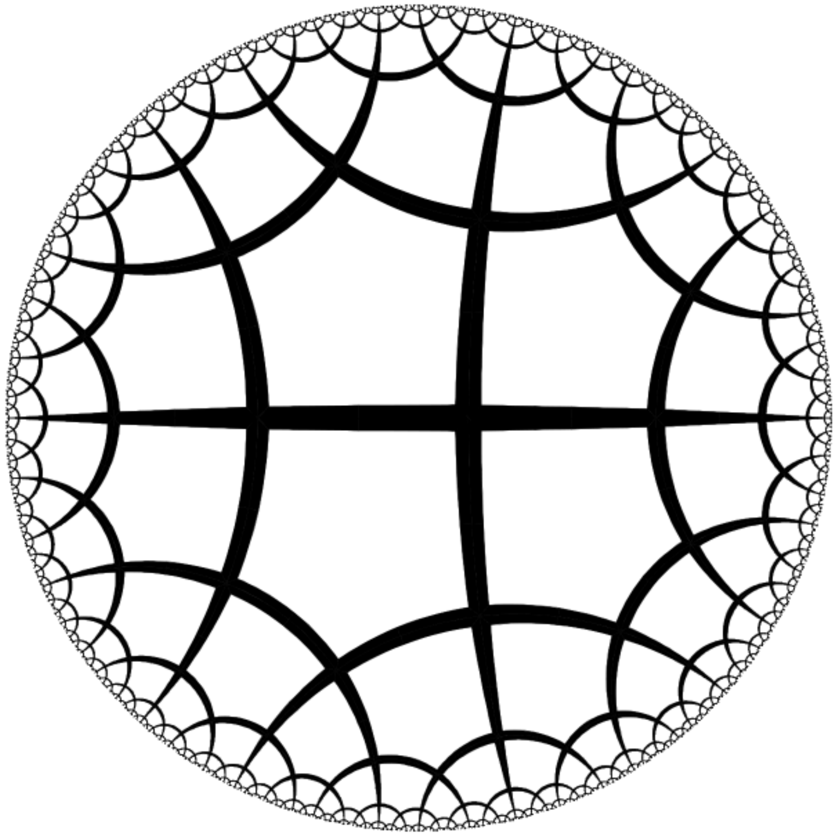

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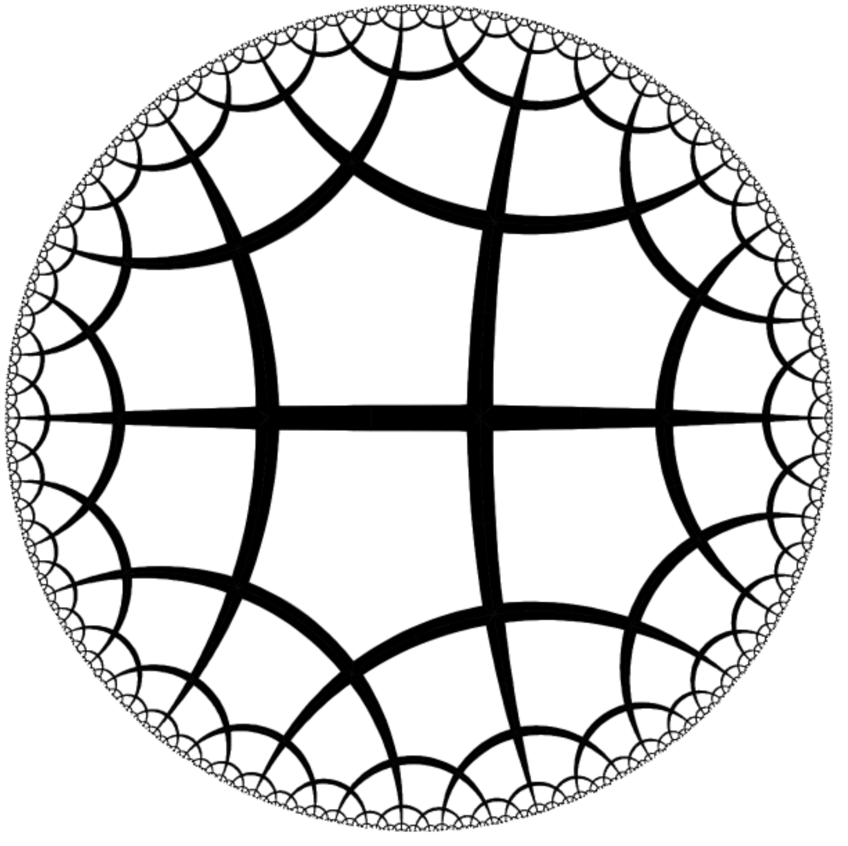

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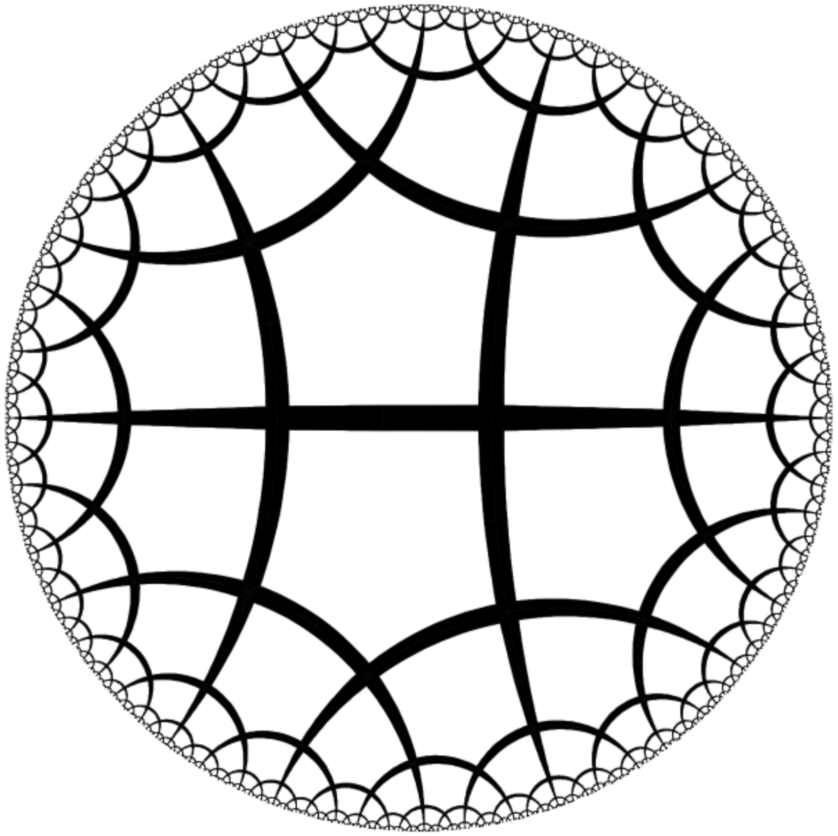

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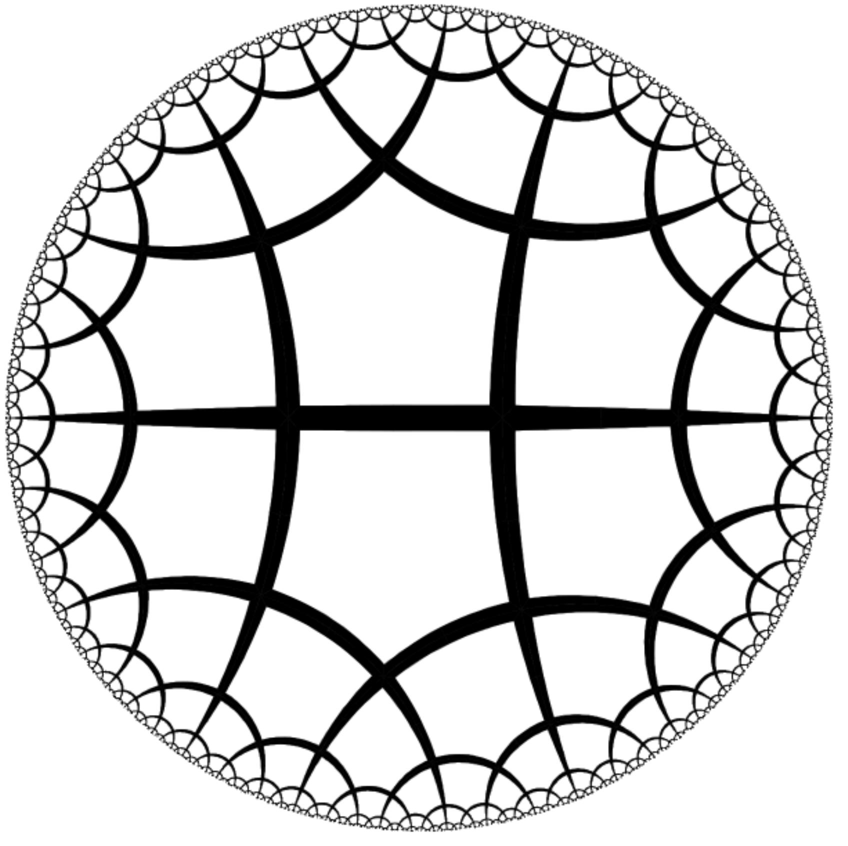

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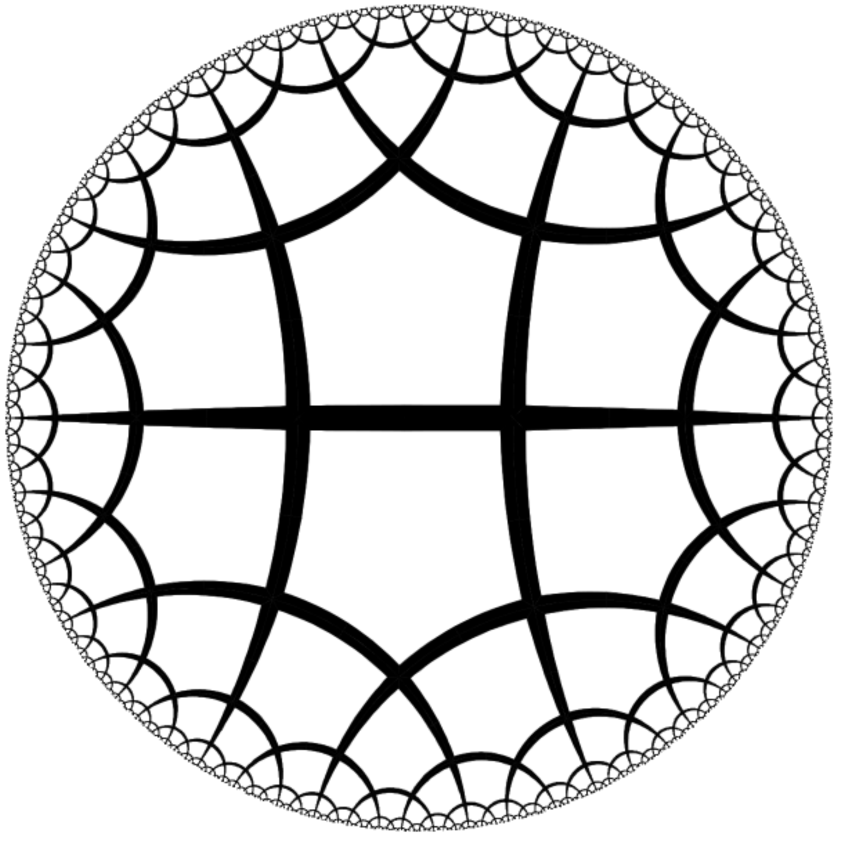

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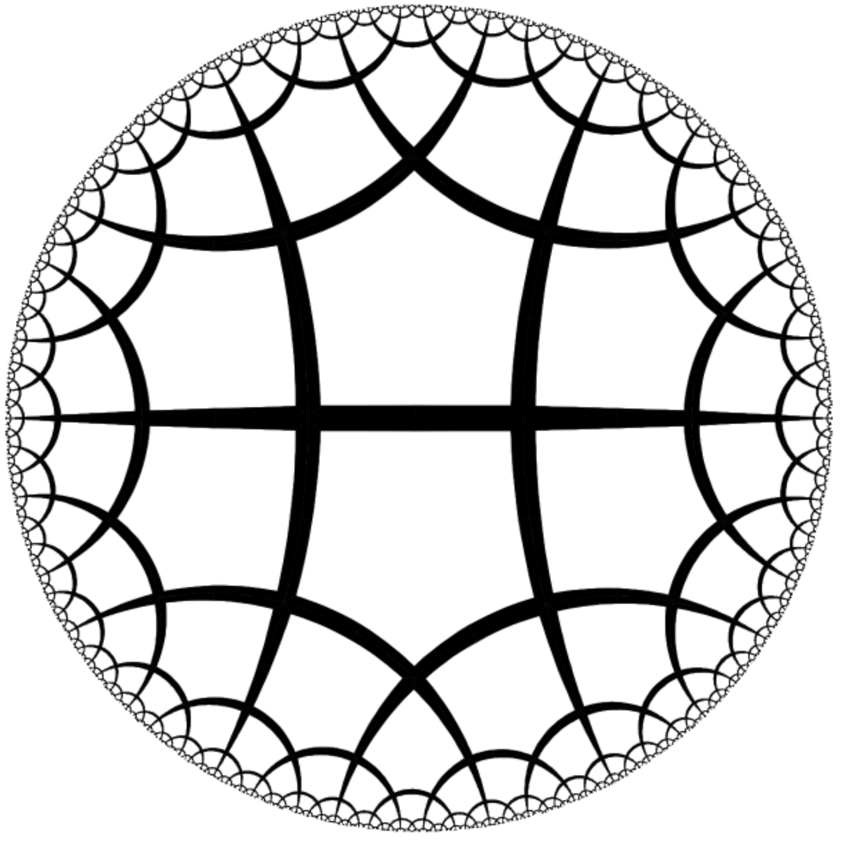

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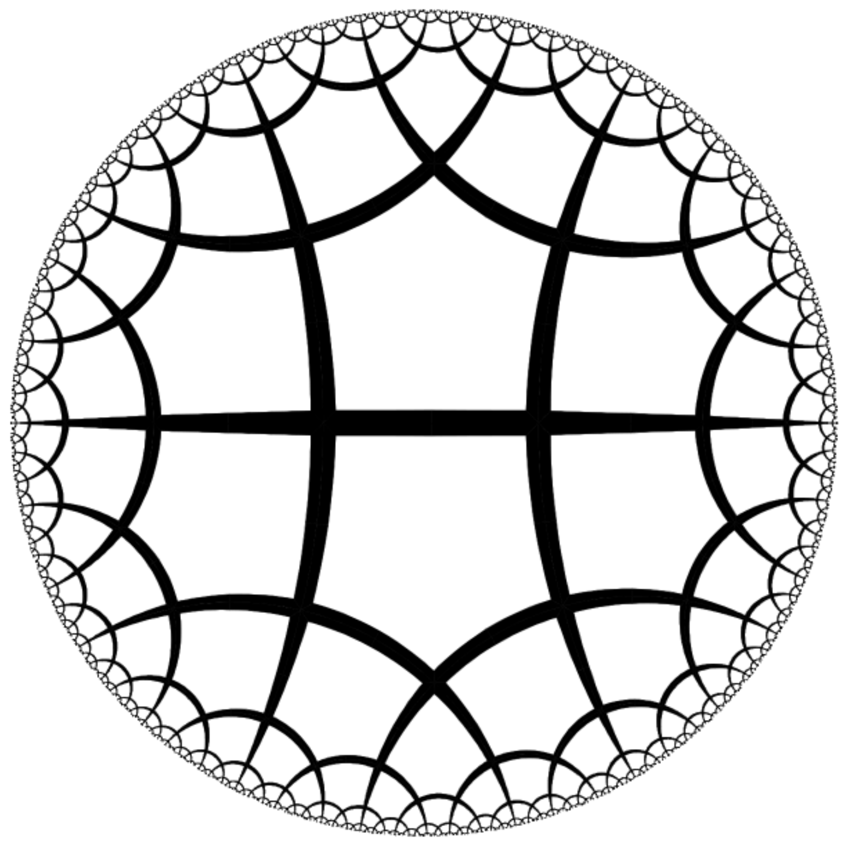

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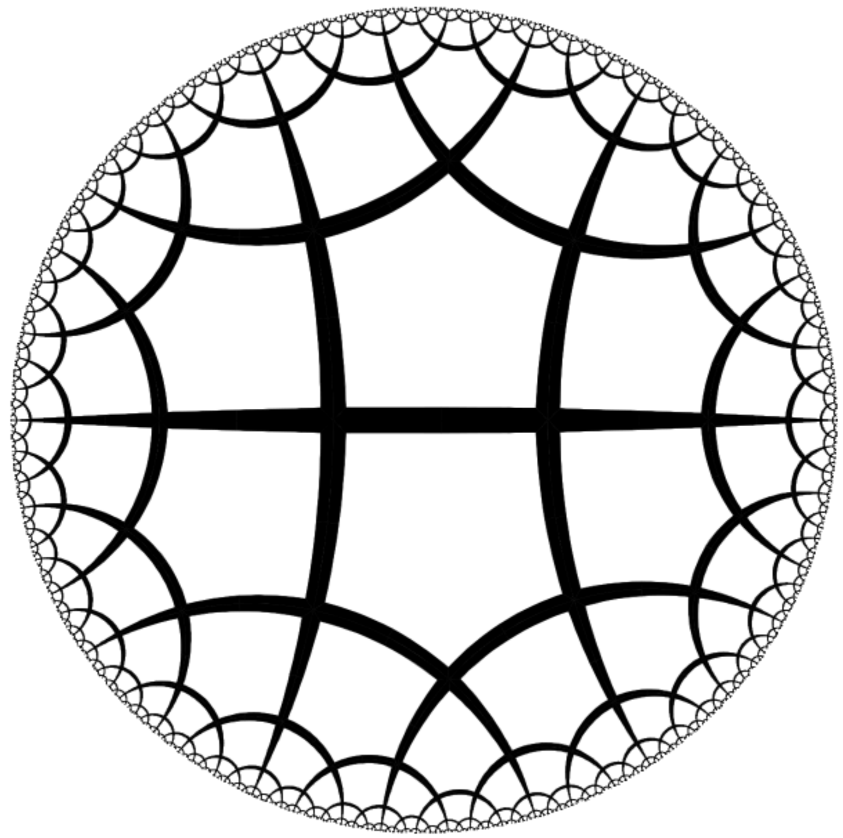

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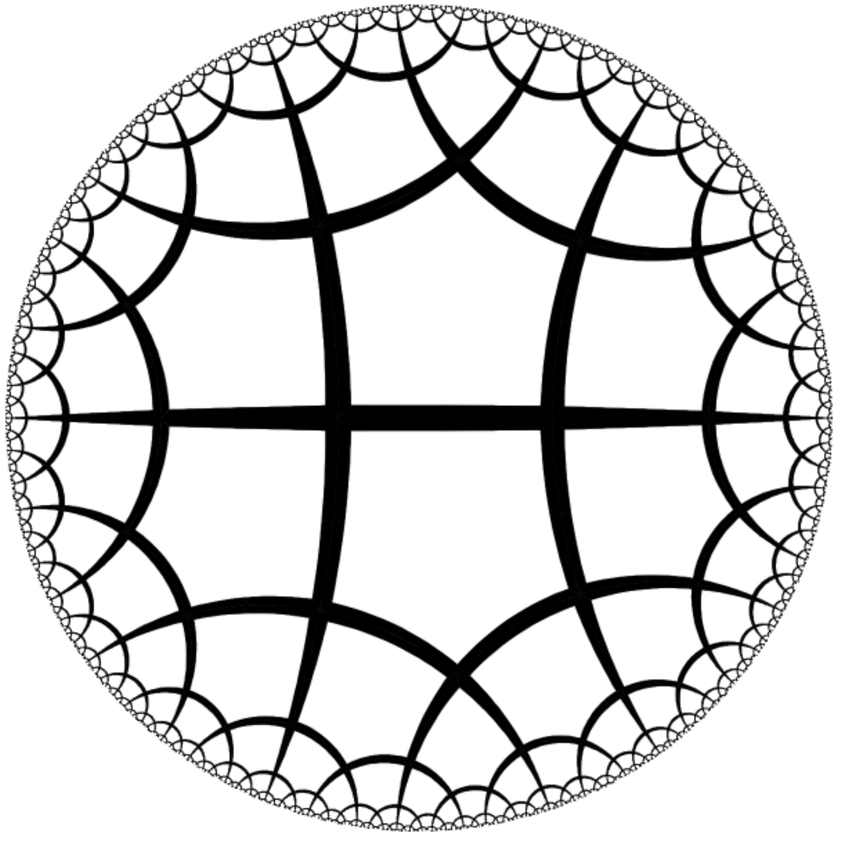

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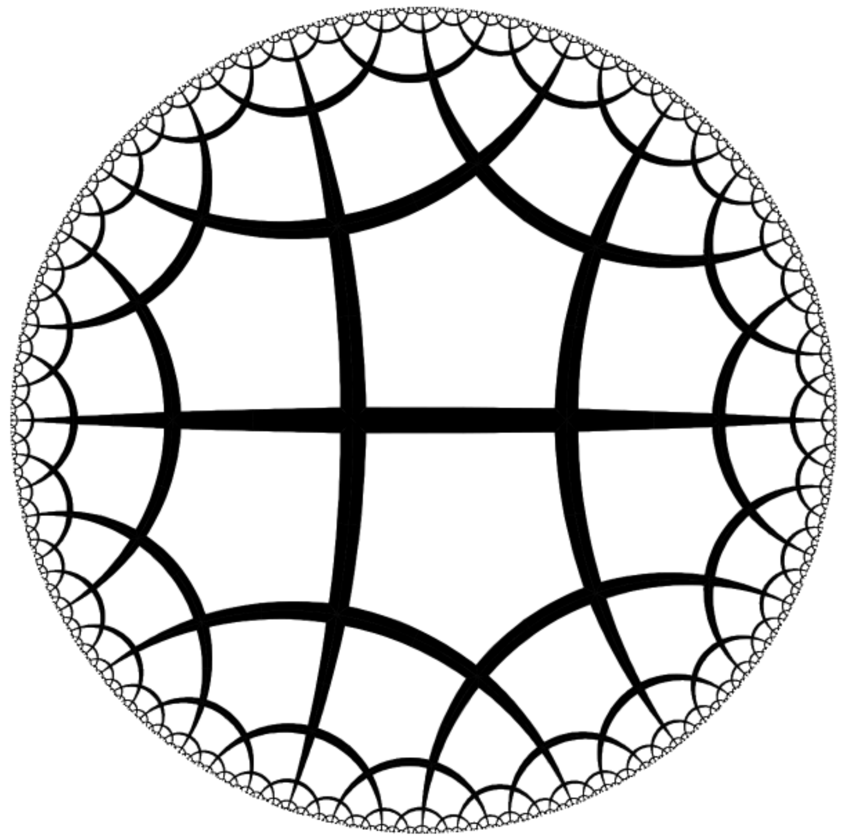

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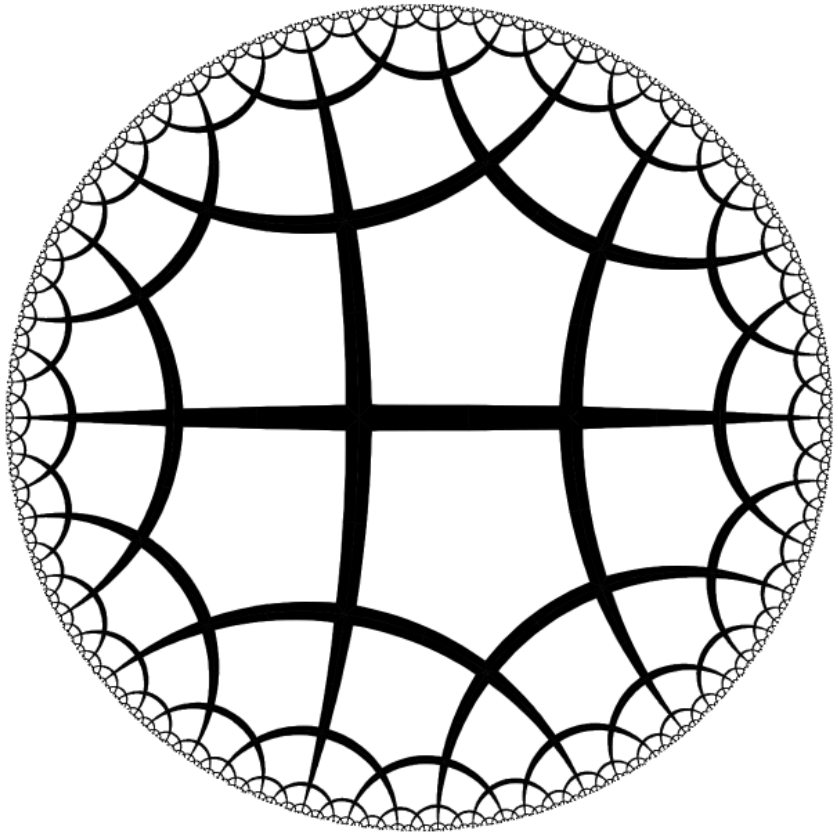

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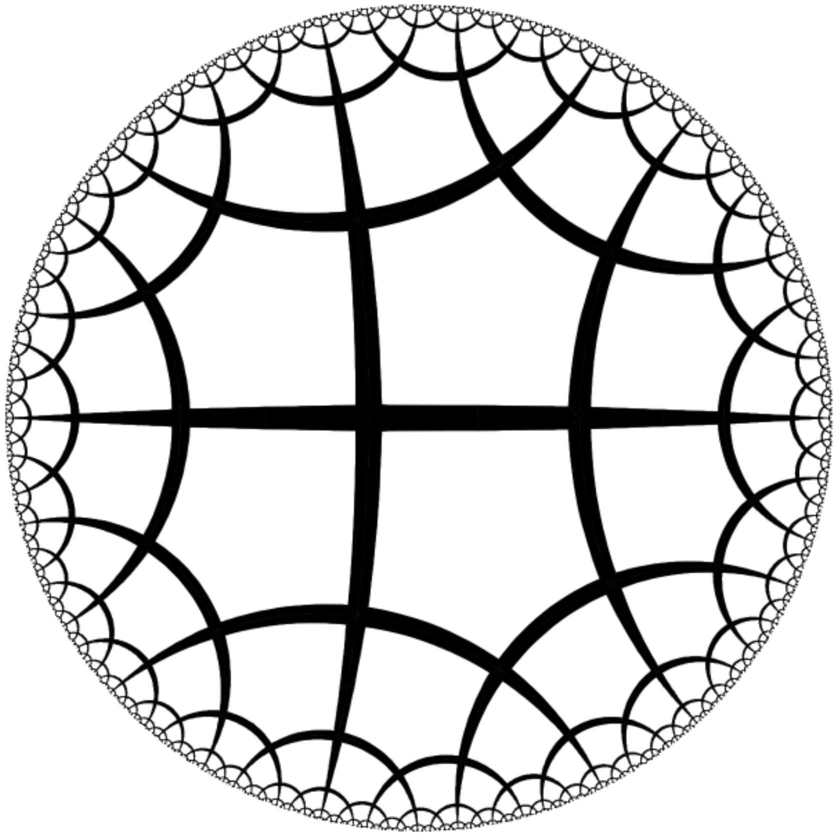

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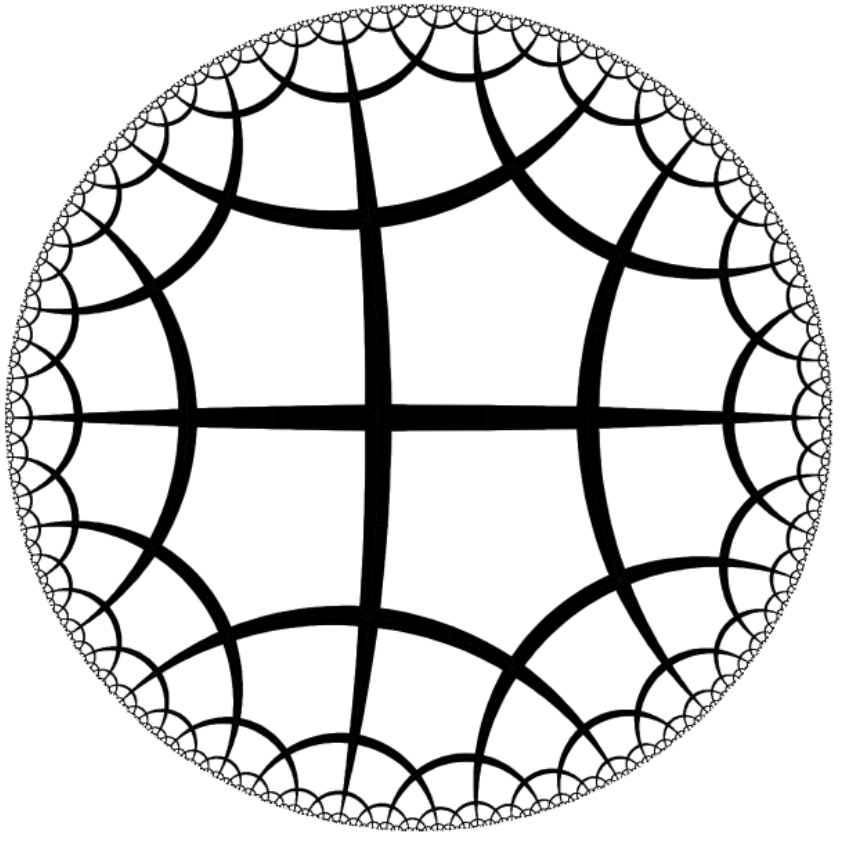

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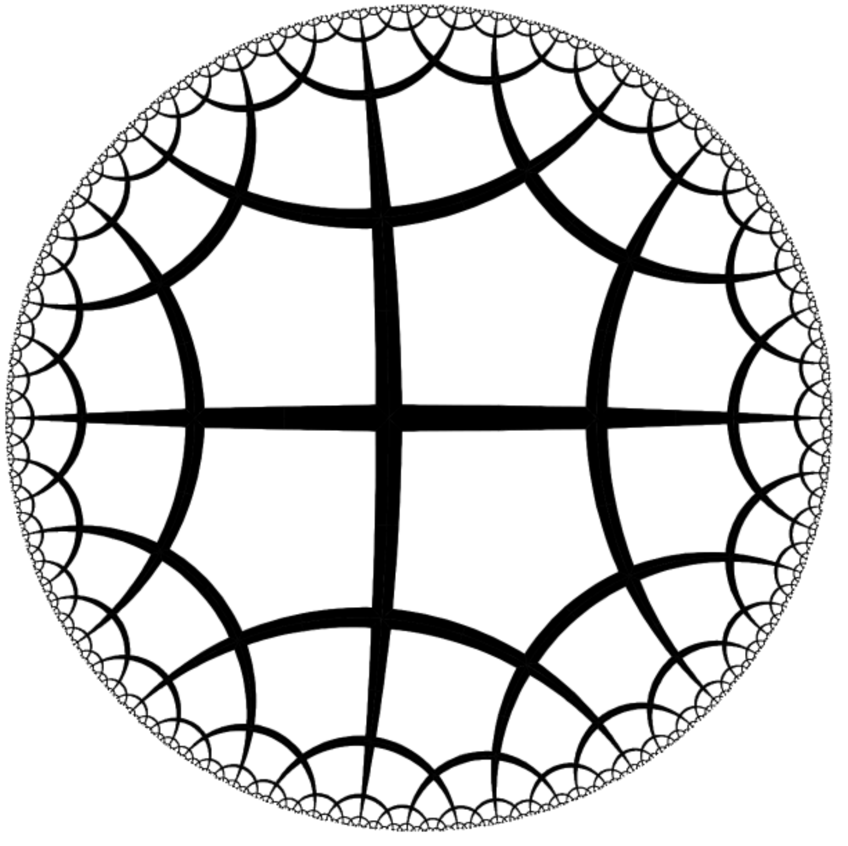

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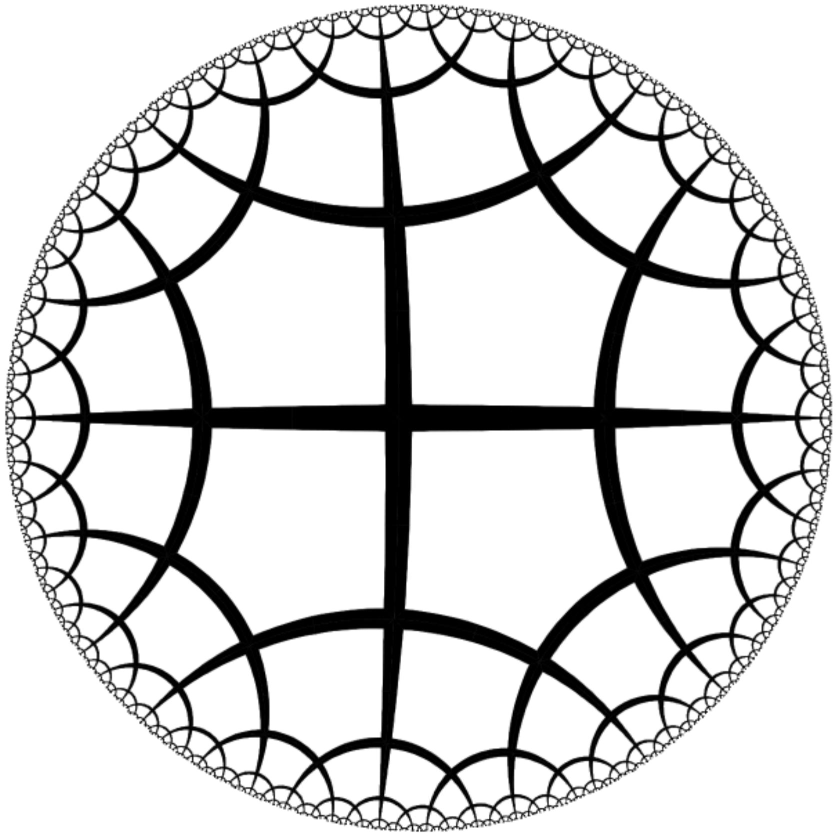

Hyperbolic isometry