Complex manifolds of dimension 1

lecture 8: Geodesics on the Poincaré plane

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Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $\mathbb{H}^n := SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined by the following lemma, proven in Lecture 3.

LEMMA: Let $M = G/H$ be a simply connected space form. **Then M admits a unique (up to a constant multiplier) G-invariant Riemannian form.**

REMARK: We shall consider space forms as Riemannian manifolds equipped with a G-invariant Riemannian form.

Upper half-plane as a Riemannian manifold (reminder)

THEOREM: Let G be a group of orientation-preserving conformal (that is, holomorphic) automorphisms of the upper halfplane $\mathbb{H}^2$. **Then $G = PSL(2,\mathbb{R})$ and the stabilizer of a point is S^1.**

REMARK: $PSL(2,\mathbb{R}) = SO(1,2)$

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a G-invariant metric.

REMARK: This metric is unique up to a constant multiplier, and **$\mathbb{H}^2 = PSL(2,\mathbb{R})/S^1$ is a hyperbolic space.**

THEOREM: Let (x,y) be the usual coordinates on the upper half-plane $\mathbb{H}^2$. **Then the Riemannian structure s on $\mathbb{H}^2$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

THEOREM: Geodesics on a Poincaré half-plane are vertical half-lines and their images under the action of $PSL(2,\mathbb{R})$.

Geodesics in Poincaré half-plane (reminder)

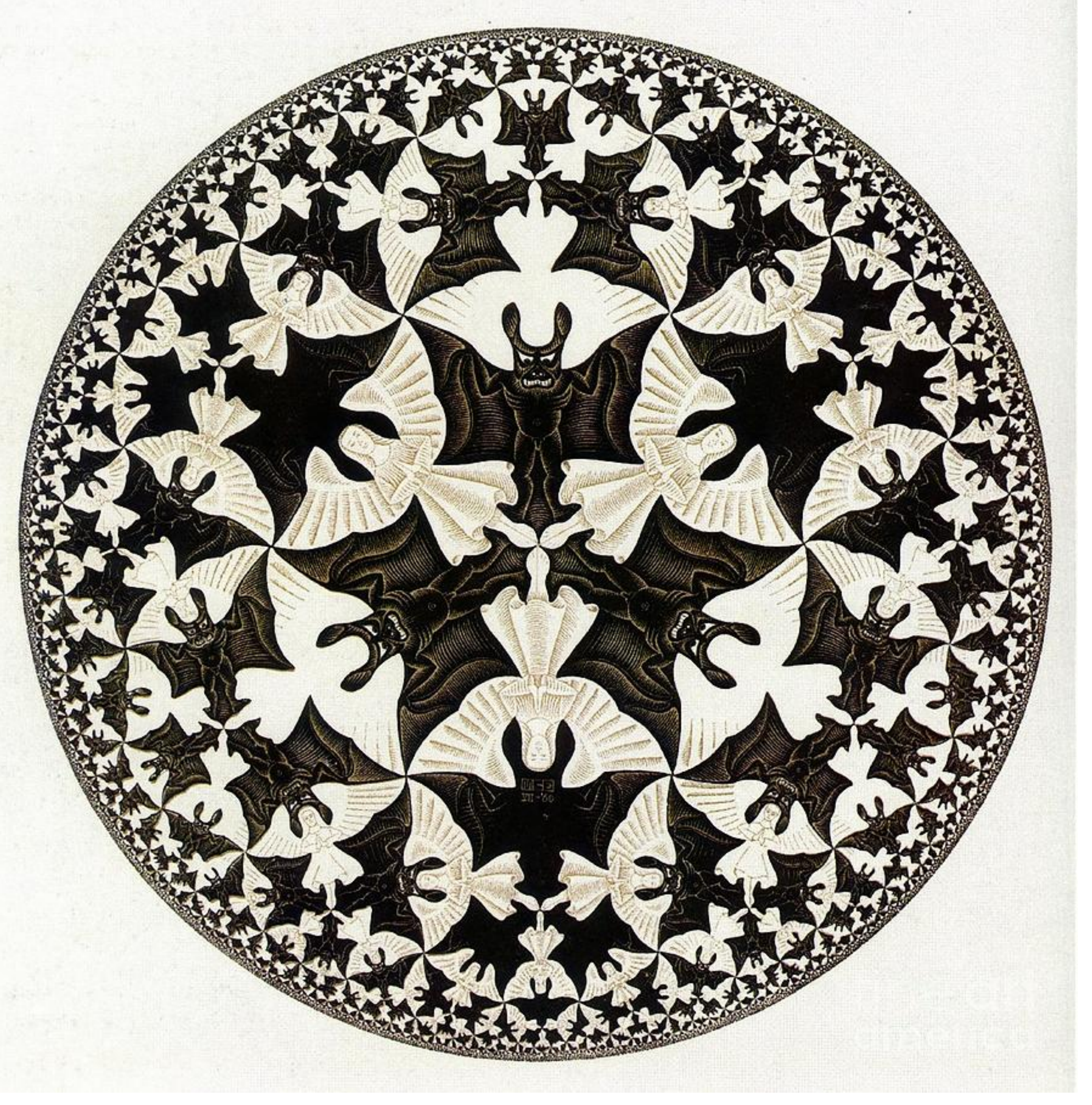
CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle.
■

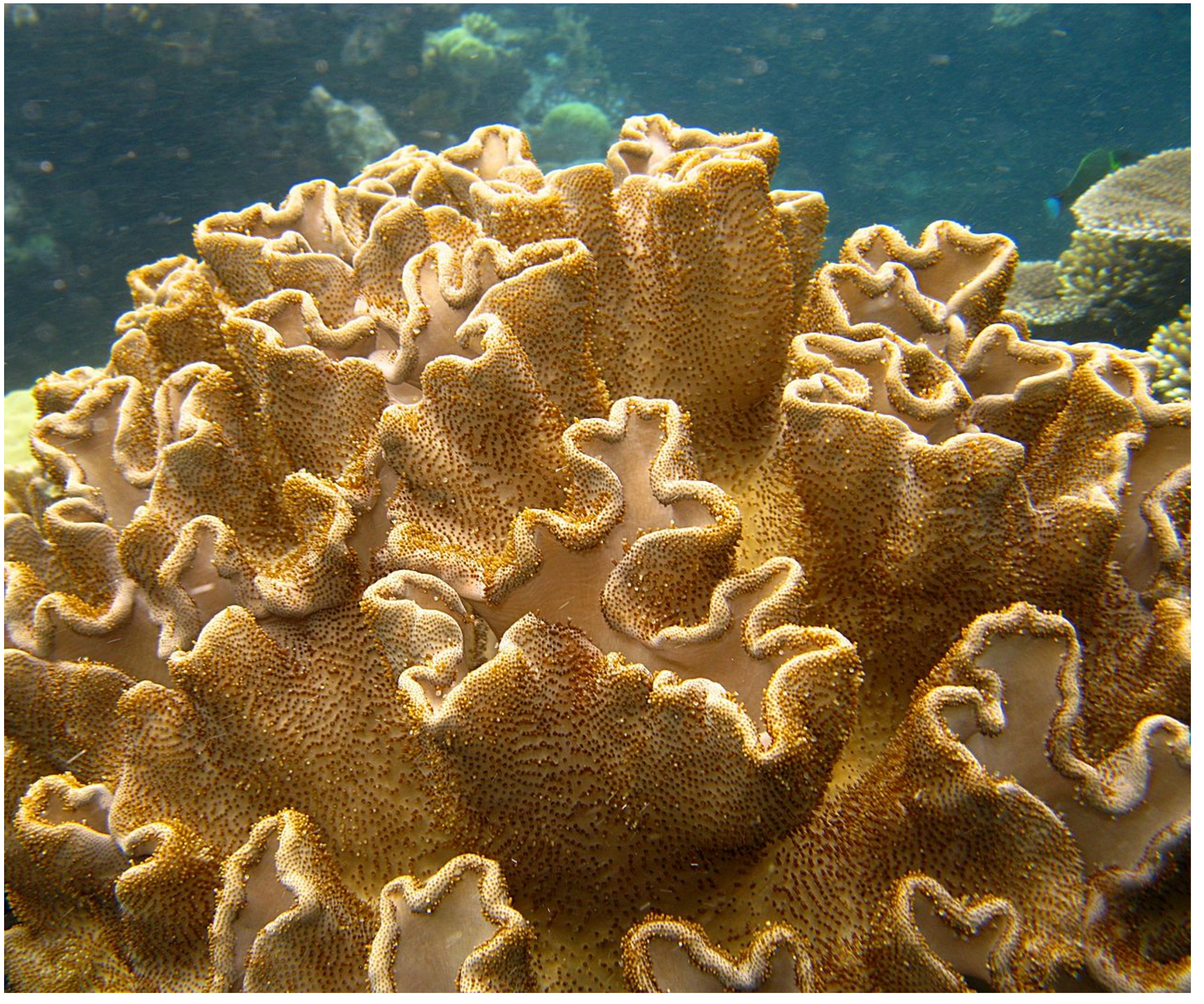
COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

M. C. Escher, Circle Limit IV



Crochet coral (Great Barrier Reef, Australia)



Reflections and geodesics

DEFINITION: A reflection on a hyperbolic plane is an involution which reverses orientation and has a fixed set of codimension 1.

EXAMPLE: Let the quadratic form q be written as $q(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2$. Then the map $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$ is clearly a reflection.

CLAIM: Fixed point set of a reflection is a geodesic. This produces **a bijection between the set of geodesics and the set of reflections.**

Proof: Let $x, y \in F$ be two distinct points on a fixed set of a reflection τ. Since the geodesic connecting x and y is unique, it is τ-invariant. Therefore, it is contained in F. **It remains to show that any geodesic on $\mathbb{H}$ is a fixed point set of some reflection.**

Let γ be a vertical line $x = 0$ on the upper half-plane $\{(x, y) \in \mathbb{R}^2, y > 0\}$ with the metric $\frac{dx^2 + dy^2}{y^2}$. Clearly, γ is a fixed point set of a reflection $(x, y) \longrightarrow (-x, y)$. **Since every geodesic is conjugate to γ, every geodesic is a fixed point set of a reflection.** ■

Geodesics on hyperbolic plane

Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), Pos $:= \{v \in V \mid q(v) > 0\}$, and $\mathbb{P}$Pos its projectivisation. Then $\mathbb{P}$Pos $= SO^+(1,2)/SO(1)$ **(check this)**, giving $\mathbb{P}$Pos $= \mathbb{H}^2$; **this is one of the standard models of a hyperbolic plane.**

REMARK: Let $l \subset V$ be **a line,** that is, a 1-dimensional subspace. The property $q(x,x) < 0$ for a non-zero $x \in l$ is written as $q(l,l) < 0$. A line l with $q(l,l) < 0$ is called **negative line**, a line with $q(l,l) > 0$ is called **positive line**.

PROPOSITION: Reflections on $\mathbb{P}$Pos **are in bijective correspondence with negative lines $l \subset V$.**

(see the proof on the next slide)

REMARK: Using the equivalence between reflections and geodesics established above, this proposition can be reformulated by saying that **geodesics on $\mathbb{P}$Pos are the same as negative lines $l \in \mathbb{P}V$.**

Geodesics on hyperbolic plane (2)

PROPOSITION: Reflections on $\mathbb{P}\operatorname{Pos}$ **are in bijective correspondence with negative lines $l \subset V$.**

Proof. Step 1: Consider an isometry τ of V which fixes x and acts as $v \longrightarrow -v$ on its orthogonal complement $v^\perp$. Since $v^\perp$ has signature (1,1), the set $\mathbb{P}\operatorname{Pos} \cap \mathbb{P}v^\perp$ is 1-dimensional and fixed by τ. We proved that τ fixes a codimension 1 submanifold in $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$, hence τ is a reflection.

It remains to show that **any reflection is obtained this way.**

Step 2: Since geodesics are fixed point sets of reflections, and all geodesics are conjugate by isometries, **all reflections are also conjugated by isometries.** Therefore, it suffices to prove that the reflection $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$ is obtained from a negative line l. Let $l = (0, 0, \lambda)$. Then $\tau(x_1, x_2, x_3) = -x_1, -x_2, x_3$, and on $\mathbb{P}V$ this operation acts as $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$. ■

REMARK: This also implies that **all geodesics in $\mathbb{P}\operatorname{Pos}$ are obtained as intersections $\mathbb{P}\operatorname{Pos} \cap \mathbb{P}W$, where $W \subset V$ is a subspace of signature (1,1).**

Geodesics and the absolute

Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $\operatorname{Pos} := \{v \in V \ \ | \ \ q(v) > 0\}$, and $\mathbb{P}\operatorname{Pos}$ its projectivisation. Then $\mathbb{P}\operatorname{Pos} = SO^+(1,2)/SO(1)$, giving $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$.

DEFINITION: A line $l \in V$ is **isotropic** if $q(l,l) = 0$. **Absolute** of a hyperbolic plane $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$ is the set of all isotropic lines,

$$\operatorname{Abs} := \{l \in \mathbb{P}V \ \ | \ \ q(l,l) = 0\}.$$

It is identified with the boundary of the disk $\mathbb{P}\operatorname{Pos} \subset \mathbb{P}V = \mathbb{R}P^2\}$.

CLAIM: Let $l \in \mathbb{P}V$ be a negative line, and $\gamma := \mathbb{P}l^\perp \cap \mathbb{P}\operatorname{Pos}$ the corresponding geodesic. **Then $l^\perp$ intersects the absolute in precisely 2 points,** called **the boundary points of γ**, or **ends of γ**. Conversely, **every geodesic is uniquely determined by the two distinct points in the absolute.**

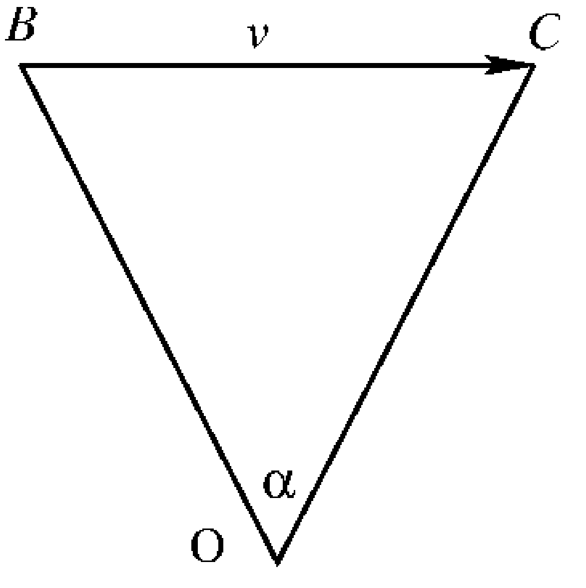
Proof: The plane $l^\perp$ has signature (1,1), and the set $q(v) = 0$ is a union of two isotropic lines in $l^\perp$. Each of these lines lies on the boundary of the set $\mathbb{P}l^\perp \cap \mathbb{P}\operatorname{Pos}$. Conversely, suppose that $\mu, \rho \in \operatorname{Abs}$ are two distinct lines. The corresponding 2-dimensional plane W has signature (1,1), because it has precisely two isotropic lines **(if it has more than two, $q|_W = 0$, which is impossible - prove it!)**. As shown above, $\mathbb{P}W \cap \mathbb{P}\operatorname{Pos}$ is a geodesic. ■

Classification of isometries of a Euclidean plane

THEOREM: Let α be a non-trivial isometry of $\mathbb{R}^2$ with Euclidean metric preserving the orientation. **Then α is either a parallel translation or a rotation with certain center on $\mathbb{R}^2$.**

Proof. Step 1: If α fixes a point $a \in \mathbb{R}^2$, then it is clearly a rotation. However, the group A of parallel translations acts transitively on $\mathbb{R}^2$, hence there exists $a \in A$ such that $a\alpha$ fixes a point on $\mathbb{R}^2$. Then $r := a\alpha$ is a rotation, and $\alpha = a^{-1}r$ is a composition of a parallel translation and rotation.

Step 2: It remains to show that a composition of a rotation a with center in A and angle α and a parallel transport R along a vector $\vec{v} \in \mathbb{R}^2$ has a fixed point. Consider a triangle ABC with $BC = \vec{v}$, $|AB| = |AC|$ and angle $\angle(BAC) = \alpha$. Clearly, aR maps C to itself. ∎



Isomorphism between $SO(1,2)$ and $PSL(2,\mathbb{R})$

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional real space equipped with a scalar product of signature (1,2), $SO^+(1,2)$ a connected component of unity, and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

CLAIM: $PSL(2,\mathbb{R}) \cong SO^+(1,2)$

Proof: Consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow$ Tr(ab). **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism ("Corollary 2" in Lecture 3). ■

The isomorphism $\mathfrak{sl}(V) = \mathrm{Sym}^2(V)$

REMARK: For any finite-dimensional vector space, one has $V \otimes V^* = \operatorname{End} V$ **(prove it).**

PROPOSITION: Let V be a 2-dimensional vector space. **Then $\mathfrak{sl}(V)$ is isomorphic to $\mathrm{Sym}^2(V)$** (the space of symmetric 2-tensors), and this isomorphism is compatible with the $SL(V)$-action.

Proof: Fix a non-degenerate 2-form ω on V. Since ω is $SL(2,\mathbb{R})$-invariant, we can use ω to construct the isomorphism $\mathfrak{sl}(V) = \mathrm{Sym}^2(V)$.

The first way to see the isomorphism $\mathfrak{sl}(V) = \mathrm{Sym}^2(V)$: Now, $V \otimes V^* = \operatorname{End} V$ and $V \otimes V = \mathrm{Sym}^2(V) \oplus \Lambda^2(V)$. Using the form ω to produce an isomorphism $V = V^*$, we find that the decomposition

$$\operatorname{End} V = V \otimes V^* = \langle \mathrm{Id}_V \rangle \oplus \mathfrak{sl}(V)$$

is identical to

$$\operatorname{End} V = V \otimes V = \langle \omega \rangle \oplus \mathrm{Sym}^2(V).$$

The isomorphism $\mathfrak{sl}(V) = \operatorname{Sym}^2(V)$ (2)

The second way to see the isomorphism $\mathfrak{sl}(V) = \operatorname{Sym}^2(V)$: Now, $A \in \mathfrak{sl}(V)$ if and only if $\omega(A(x), y) = -\omega(x, A(y))$. Indeed, $A \in \mathfrak{sl}(V)$ if and only if $e^A \in SL(V)$, and this gives $\omega(e^{tA}(x), e^{tA}y) = \omega(x, y)$. Taking derivative in t, we obtain $\omega(A(x), y) = -\omega(x, A(y))$. However, $\omega(A(x), y) = \omega(A(y), x)$, hence $A \in \mathfrak{sl}(V)$ if and only if $\omega(A(x), y)$ is a symmetric 2-form. ■

Corollary 1: Let $A \in SL(2, \mathbb{R})$ be a matrix with eigenvalues α, α^{-1}, and $B \in SO(1,2)$ the endomorphism associated with A through $PSL(2, \mathbb{R}) \cong SO^+(1,2)$. **Then B has eigenvalues $\alpha^2, 1, \alpha^{-2}$.**

Proof: Let x, y be a basis in V. Then $x^2 = x \otimes x, xy = x \otimes y, y^2 = y \otimes y$ is a basis in $\operatorname{Sym}^2(V)$. When x, y are eigenvectors of A with eigenvalues α, α^{-1}, the tensors x^2, xy, y^2 are eigenvectors for B with eigenvalues $\alpha^2, 1, \alpha^{-2}$. ■

Classification of isometries of a hyperbolic plane (part 1)

THEOREM: Let $A \in SL(2,\mathbb{R})$, and $\alpha \in SO^+(1,2)$ the corresponding isometry of a hyperbolic plane. Denote by q the quadratic form of signature $(1,2)$ on R^3. Assume that $\alpha \neq \mathrm{Id}$, that is, $A \neq \pm 1$. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$. In this case α is called **"elliptic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| < 2$; it is conjugate to a rotation of a disk around 0.

(ii) α has an eigenvector x with $q(x,x) < 0$. In that case α is called **"hyperbolic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| > 2$; it is conjugate to a matrix $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, with $t \neq \pm 1$.

(iii) α has a unique eigenvector x with $q(x,x) = 0$. In that case α is called **"parabolic isometry"**. The matrix A satisfies $|\operatorname{Tr} A| = 2$, and is conjugate to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

Proof in the next lecture