Complex manifolds of dimension 1

lecture 9: Kobayashi pseudometric and Arzela-Ascoli theorem

Misha Verbitsky

IMPA, sala 232

January 24, 2020

Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Let $G = PSL(2,\mathbb{R})$ be a group of orientation-preserving conformal (that is, holomorphic) automorphisms of the upper halfplane $\mathbb{H}^2$. **Poincaré half-plane** is the upper half-plane equipped with a G-invariant metric of constant negative curvature constructed in Lecture 7.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}^2$. **Then the Riemannian structure s on $\mathbb{H}^2$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

DEFINITION: **Minimising geodesic** in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

THEOREM: **Geodesics on a Poincaré half-plane are vertical half-lines and their images under the action of $PSL(2,\mathbb{R})$.**

Geodesics in Poincaré half-plane

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle.
■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

Poincaré metric on a disk

DEFINITION: Poincaré metric on the unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier; prove it)**.

DEFINITION: Let $f: M \longrightarrow M_1$ be a map of metric spaces. Then f is called **C-Lipschitz** if $d(x, y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential, taken with respect to the Poincaré metric, satisfies $|D\varphi_x|_P \leqslant 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■

Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y) := \sup_{d\in\mathfrak{D}} d(x,y)$ is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

EXAMPLE: The Kobayashi pseudometric on $\mathbb{C}$ vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ■

Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipshitz. ■

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

Caratheodory metric

DEFINITION: Let $x, y \in M$ be points on a complex manifold. Define **Caratheodory pseudometric** as $d_C(x,y) = \sup\{d_P(f(x), f(y))\}$, where the supremum is taken over all holomorphic map $f: M \longrightarrow \Delta$, and d_P is Poincare metric on the disk Δ.

REMARK: Usually the term "Kobayashi/Caratheodory pseudometric" is abbreviated to "Kobayashi/Caratheodory metric", **even when it is not a metric.**

REMARK: Caratheodory pseudometric **satisfies the triangle inequality** because a supremum of pseudometrics satisfies triangle inequality.

EXERCISE: Prove that **Caratheodory pseudometric is bounded by the Kobayashi pseudometric:** $d_K \geqslant d_C$.

REMARK: Clearly, $d_C \neq 0$ on any bounded domain.

Complex hyperbolic space

DEFINITION: Let $V = \mathbb{C}^{n+1}$ be a complex vector space equipped with a Hermitian metric h of signature $(1, n)$, and $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}V$ projectivization of the set of positive vectors $\{x \in V \quad h(x, \overline{x}) > 0\}$. Then $\mathbb{H}^n_{\mathbb{C}}$ is equipped with a homogeneous action of $U(1, n)$. The same argument as used for space forms implies that $\mathbb{H}^n_{\mathbb{C}}$ admits a $U(1, n)$-invariant Hermitian metric, which is unique up to a constant multiplier. This Hermitian complex manifold is called **complex hyperbolic space**.

REMARK: For $n > 1$ **it is not isometric to the real hyperbolic spaces defined earlier.**

REMARK: As a complex manifold $\mathbb{H}^n_{\mathbb{C}}$ **is isomorphic to an open ball in** $\mathbb{C}^n$ **(prove it!)**

REMARK: The Kobayashi metric and the Caratheodory metric on $\mathbb{H}^n_{\mathbb{C}}$ are $U(1, n)$-invariant, because $U(1, n)$ acts holomorphically, hence proportional to the hyperbolic metric, which is also called **Bergman metric** on an open ball.

EXERCISE: Prove that **Kobayashi metric on a ball in** $\mathbb{C}^n$ **is equal to the Caratheodory metric.**

Uniform convergence for Lipschitz maps

DEFINITION: A sequence of maps $f_i: M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f: M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i: M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). Then there exists N such that $\sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$. Since f_i are 1-Lipschitz, this implies that

$$\sup_{y\in K} d(f_{N+i}(y), f(y)) \leqslant$$
$$\leqslant d(f_{N+i}(x), f(x)) + (d(f_{N+i}(x), f_{N+i}(y)) + d(f(x), f(y)) \leqslant 3\varepsilon,$$

where $x \in N_\varepsilon$ is chosen in such a way that $d(x, y) < \varepsilon$. ■

EXERCISE: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Arzela-Ascoli theorem for Lipschitz maps

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha: M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite set of 1-Lipschitz maps $f_\alpha: M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f: M \longrightarrow \mathbb{C}$ uniformly on compacts.**

REMARK: The limit f is clearly also 1-Lipschitz.

Proof. Step 1: Suppose we can prove Arzela-Ascoli when M is compact. Then we can choose a sequence of compact subsets $K_i \subset M$, find subsequences in $\mathcal{F}$ converging on each K_i, and use the diagonal method to find a subsequence converging on all K_i. Therefore, **we can assume that M is compact.**

Arzela-Ascoli theorem for Lipschitz maps (2)

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

Proof. Step 1: We can assume that M is compact, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.

Step 2. By definition of pointwise convergence, for any finite set $S \subset M$, there exists a subsequence f_i of $\mathcal{F}$ which converges to $f \in \operatorname{Map}(S, N)$ in S. Using diagonal method, we choose a subsequence f_i of $\mathcal{F}$ which converges to $f \in \operatorname{Map}(M', N)$ pointwise in a dense countable set $M' \subset M$. **Then f_i converges to f uniformly by Claim 1.** ■