Complex manifolds of dimension 1

lecture 10: Riemann mapping theorem

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Uniform convergence for Lipschitz maps (reminder)

DEFINITION: A sequence of maps $f_i : M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). Then there exists N such that $\sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$. Since f_i are 1-Lipschitz, this implies that

$$\sup_{y\in K} d(f_{N+i}(y), f(y)) \leqslant$$
$$\leqslant d(f_{N+i}(x), f(x)) + (d(f_{N+i}(x), f_{N+i}(y)) + d(f(x), f(y)) \leqslant 3\varepsilon,$$

where $x \in N_\varepsilon$ is chosen in such a way that $d(x, y) < \varepsilon$. ■

COROLLARY 1: The space of Lipschitz maps **is closed in the topology of pointwise convergence.** Moreover, **pointwise convergence of Lipschitz maps implies uniform convergence on compacts.**

Arzela-Ascoli theorem for Lipschitz maps

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha: M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite set of 1-Lipschitz maps $f_\alpha: M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f: M \longrightarrow \mathbb{C}$ uniformly on compacts.**

REMARK: **The limit f is clearly also 1-Lipschitz.**

REMARK: It was proven in Lecture 9.

Arzela-Ascoli theorem for Lipschitz maps (a second proof)

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

Proof. Step 1: Using the diagonal argument, **we can assume that M is compact, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.** It remains to show that the space of Lipschitz maps from M to N is compact with topology of uniform convergence.

Step 2. The space of maps to a compact is compact in topology of point-wise convergence **(Tychonoff theorem)**. However, **on Lipschitz maps, pointwise convergence implies uniform convergence** (Corollary 1). ■

Normal families of holomorphic functions

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzela-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzela-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected, bounded domain. **Then Ω is biholomorphic to Δ.**

Idea of a proof: We consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df_x|$ maximal in the sense of Kobayashi metric in the closure of $\mathcal{F}$. **Such f exists by Montel's theorem. We prove that f is a bijective isometry**, and hence biholomorphic.

Functions which take distinct values on a boundary of a disk

LEMMA: Let u, v be non-constant holomorphic functions on a disk Δ, continuously extended to its boundary, and $u_t(z) = u(tz)$, $v_t(z) = u(tz)$, where $t \in]0,1]$. **Then for some $s, t \in]0,1]$, $u_s(z) \neq v_t(z)$ for all $z \in \partial\Delta$.**

Proof. Step 1: Consider the function $z \longrightarrow u(z) - v(z)$. Unless $u = v$, this function has finitely many zeros on a compact disk. Choose $s = t$ in such a way that the boundary of a circle of radius t with center in 0 avoids all these zeros. **Then $u_t(z) \neq v_t(z)$ on $z \in \partial\Delta$.**

Step 2: Now, if $u = v$, we replace u by u_r, for some $r \in]0,1]$. Unless $u = const$, **this is a different holomorphic function.** Now we can apply the previous argument, obtaining functions u_{rt} and v_t which satisfy $u_{rt}(z) \neq v_t(z)$ for all $z \in \partial\Delta$. ■

The set of injective holomorphic maps is closed

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps and constant maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

Proof: Let f_i be a sequence of injective maps converging to $f: \Omega_1 \longrightarrow \Omega_2$ which is not injective. Then $f(a) = f(b)$ for some $a \neq b$ in Ω_1. Choose open disks A and B containing a and b. Using the previous lemma, we may shrink A and B, and identify A and B in such a way that the functions g and h obtained by restricting f to $\partial A = \partial B$ are non-equal everywhere on the boundary. Then Proposition is implied by the following lemma.

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct, non-constant holomorphic functions $g, h: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $h(x) = g(x)$ for some $x \in \Delta$, but $h(x) \neq g(x)$ everywhere on the boundary. **Then $\mathcal{R}$ is open in uniform topology.**

The set of non-injective, non-constant maps is open

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct, non-constant holomorphic functions $g, h: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $h(x) = g(x)$ for some $x \in \Delta$, but $h(x) \neq g(x)$ everywhere on the boundary. **Then $\mathcal{R}$ is open in uniform topology.**

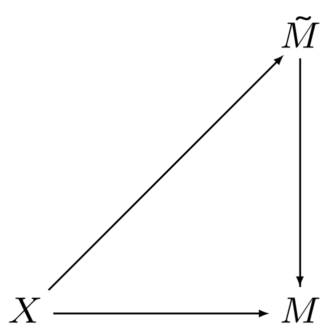
Proof. Step 1: Consider the function $\frac{(h-g)'}{h-g}$ on Δ. **This function has a simple pole in all the points where $h = g$.** Moreover, $n_{h,g} := \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} dz$ is equal to the number of points $x \in \Delta$ such that $h(x) = g(x)$ (taken with multiplicities, which are always positive integers).

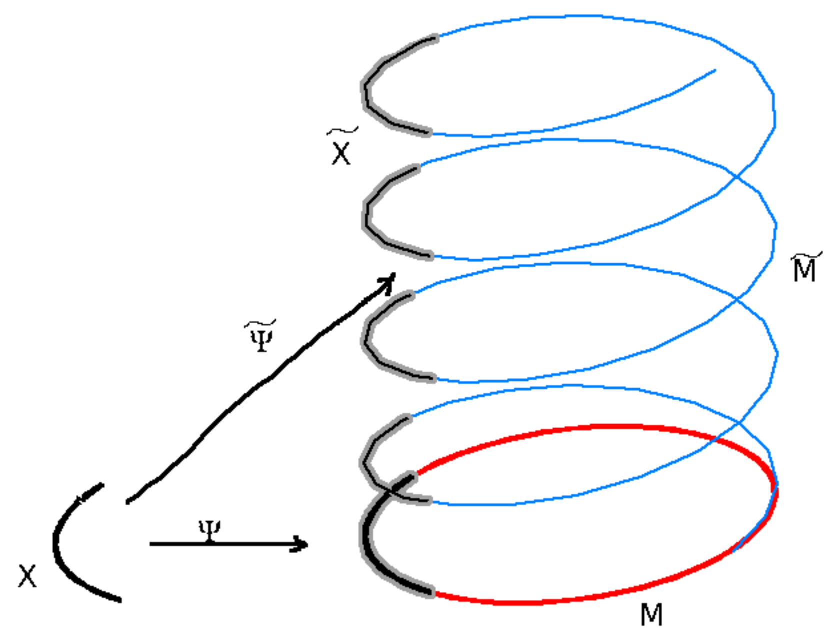
Step 2: Since the integral is continuous in unform topology, **this number is locally constant on the space of pairs such $h, g: \Delta \longrightarrow \mathbb{C}$.** Therefore, the set $\mathcal{R}$ of all h, g with $n_{h,g} \neq 0$ is open. ■

Coverings (reminder)

DEFINITION: A topological space X is **locally path connected** if for each $x \in X$ and each neighbourhood $U \ni x$, there exists a smaller neighbourhood $W \ni x$ which is path connected.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.

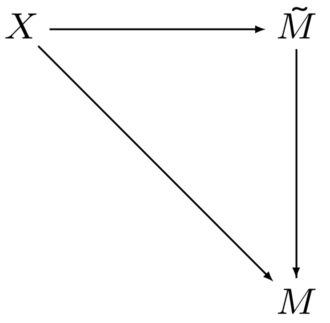




Homotopy lifting principle (reminder)

EXAMPLE: The map $x \longrightarrow x^2$ is a covering from $\mathbb{C}^* := \mathbb{C}\backslash 0$ to itself **(prove it)**.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.



COROLLARY: Let $\varphi: \Omega \longrightarrow \mathbb{C}^*$ be a holomorphic map from a simply connected domain Ω. **Then there exists a holomorphic map $\varphi_1: \Omega \longrightarrow \mathbb{C}^*$ such that for all $z \in \Delta$, $\varphi(z) = \varphi_1(z)^2$.**

Proof: We apply homotopy lifting principle to $X = \Omega$, $M = \tilde{M} = \mathbb{C}^*$, and $\tilde{M} \longrightarrow M$ mapping x to x^2. ■

REMARK: We denote $\varphi_1(z)$ by $\sqrt{\varphi(z)}$, for obvious reasons.

Poincaré metric and the map $x \longrightarrow x^2$

CLAIM: Consider a non-bijective holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from Poincare disk to itself. **Then $|d\varphi| < 1$ at each point,** where $d\varphi$ is a norm of an operator $d\varphi: T_x\Delta \longrightarrow T_{\varphi(x)}\Delta$ taken with respect to the Poincare metric.

Proof: Let $\varphi: \Delta \longrightarrow \Delta$ be a holomorphic map which satisfies $|d\varphi| = 1$ at $x \in \Delta$. Replacing φ by $\gamma_1 \circ \varphi \circ \gamma_2$ if necessary, where γ_i are biholomorphic isometries of Δ, we may assume that $x = 0$ and $\varphi(x) = 0$. By Schwartz lemma, for such φ, relation $|d\varphi(0)| = 1$ implies that φ is a linear biholomorphic map. ■

REMARK: We will apply this claim only to the function $x \xrightarrow{\varphi} x^2$. However, **even for this function it takes some work,** because an explicit proof needs and explicit form of Poincaré metric on a disk, which we did not have.

Poincaré metric and $\sqrt{\varphi}$

Corollary 2: Let $\varphi: \Delta \longrightarrow \Delta\backslash 0$ be a holomorphic function, and $\sqrt{\varphi}$ a holomorphic function defined above. Let $|d\varphi|(x)$ denote the norm of the operator $d\varphi$ at $x \in \Delta$ computed with respect to the Poincare metric on Δ. **Then $|d\varphi|(x) < |d\sqrt{\varphi}|(x)$ for any $x \in \Delta$.**

Proof: Let $\psi(x) = x^2$. By the claim above, $|d\psi|(x) < 1$ for all $x \in \Delta$ (here **the norm is taken with respect to Poincaré metric**). Using the chain rule, we obtain that $d\varphi = d\psi \circ d\sqrt{\varphi}$. which gives $|d\varphi|(x) = |d\psi|(\sqrt{\varphi}(x))|d\sqrt{\varphi}|(x)$, hence

$$|d\sqrt{\varphi}|(x) = \frac{|d\varphi|(x)|}{|d\psi|(\sqrt{\varphi}(x))} > |d\varphi|(x).$$

■

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Proof. Step 1: Consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df|(x)$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem.** Since f lies in the closure of $\mathcal{F}$, and the set of injective maps is closed, **f is injective.**

Step 2: It remains to show that f is surjective. Suppose it is not surjective: $z \notin f(\Omega)$. Taking a composition of f and an isometry of the Poincare disk does not affect $|df|(x)$, hence we may assume that $z = 0$. **Then the function $\sqrt{f}$ is a well defined holomorphic map from Ω to Δ.** By Corollary 2, $|d\sqrt{f}|(x) > |df|(x)$, which is impossible, because it $|df|(x)$ is maximal. ∎