Complex manifolds of dimension 1

lecture 11: Fatou and Julia stes

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Functions which take distinct values on a boundary of a disk

Proposition which was missing a detail in Lecture 10.

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

LEMMA: Let u, v be non-constant holomorphic functions on a disk Δ, continuously extended to its boundary, and $u_t(z) = u(tz)$, $v_t(z) = u(tz)$, where $t \in]0,1]$. **Then for some $s,t \in]0,1]$, $u_s(z) \neq v_t(z)$ for all $z \in \partial\Delta$.**

Proof. Step 1: Consider the function $z \longrightarrow u(z) - v(z)$. Unless $u = v$, this function has finitely many zeros on a compact disk. Choose $s = t$ in such a way that the boundary of a circle of radius t with center in 0 avoids all these zeros. **Then $u_t(z) \neq v_t(z)$ on $z \in \partial\Delta$.**

Step 2: Now, if $u = v$, we replace u by u_r, for some $r \in]0,1]$. Unless $u = const$, **this is a different holomorphic function.** Now we can apply the previous argument, obtaining functions u_{rt} and v_t which satisfy $u_{rt}(z) \neq v_t(z)$ for all $z \in \partial\Delta$. ∎

The set of injective holomorphic maps is closed

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \ \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

Proof: Let f_i be a sequence of injective maps converging to $f: \ \Omega_1 \longrightarrow \Omega_2$ which is not injective. Then $f(a) = f(b)$ for some $a \neq b$ in Ω_1. Choose open disks A and B containing a and b. Using the previous lemma, we may shrink A and B, and identify A and B in such a way that the functions g and h obtained by restricting f to $\partial A = \partial B$ are non-equal everywhere on the boundary. Then Proposition is implied by the following lemma.

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct, non-constant holomorphic functions $g, h: \ \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $h(x) = g(x)$ for some $x \in \Delta$, but $h(x) \neq g(x)$ everywhere on the boundary. **Then $\mathcal{R}$ is open in uniform topology.**

The set of non-injective, non-constant maps is open

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct, non-constant holomorphic functions $g, h : \quad \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $h(x) = g(x)$ for some $x \in \Delta$, but $h(x) \neq g(x)$ everywhere on the boundary. **Then $\mathcal{R}$ is open in uniform topology.**

Proof. Step 1: Consider the function $\frac{(h-g)'}{h-g}$ on Δ. **This function has a simple pole in all the points where $h = g$.** Moreover, $n_{h,g} := \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} dz$ is equal to the number of points $x \in \Delta$ such that $h(x) = g(x)$ (taken with multiplicities, which are always positive integers).

Step 2: Since the integral is continuous in unform topology, **this number is locally constant on the space of pairs such $h, g : \quad \Delta \longrightarrow \mathbb{C}$.** Therefore, the set $\mathcal{R}$ of all h, g with $n_{h,g} \neq 0$ is open. ■

Fatou and Julia sets

DEFINITION: Let X, Y be complex varieties, and $\mathcal{F}$ a family of holomorphic maps $f_\alpha : X \longrightarrow Y$. Recall that $\mathcal{F}$ is **a normal family** if any sequence $\{f_i\}$ in $\mathcal{F}$ has a subsequence which converges (uniformly on compacts) to a holomorphic map.

DEFINITION: Let $f : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rational map, and $\{f^i\} = \{f, f \circ f, f \circ f \circ f, ...\}$ the set of all iterations of f. **Fatou set of f** is the set of all points $x \in \mathbb{C}$ such that for some neighbourhood $U \ni x$, the restriction $\{f^i|_U\}$ is a normal family, with all f^i except finitely many taking values in a disk $B \subset \mathbb{C}P^1$, and **Julia set** is a complement to Fatou set.

EXAMPLE: For the map $f(x) = x^2$, Julia set is the unit circle, and the Fatou set is its complement **(prove it)**.

Attractor points

DEFINITION: Attractor point z is a fixed point of f such that $|df|(z) < 1$; the **attractor basin** for z is the set of all $x \in \mathbb{C}$ such that $\lim_i f^i(x) = z$.

CLAIM: For any fixed point z, **its attractor basin belongs to the Fatou set.**

Proof: Indeed, since $\lim_i f^i(x) = z$ for any point in attractor basin U, $\{f^i\}$ is a normal family on U **(pointwise convergence is equivalent to uniform convergence for bounded holomorphic functions by Arzela-Ascoli theorem).** ■

Clearly, z **is an attracting fixed point of** $f(t)$ **if** $f(z) = z$ **and** $|f'(z)| < 1$**.**

Newton iteration method

DEFINITION: Newton iteration for solving the polynomial equation $g(z) = 0$: a solution is obtained as a limit $\lim_i f^i(z)$, where $f(z) = z - \frac{g(z)}{g'(z)}$.

CLAIM: The solutions of $g(z) = 0$ are attracting fixed points of f.

Proof. Step 1: Let z be a solution of $g(z) = 0$, and m the multiplicity of zero of g in z. Since $g'(z)$ haz zero of multiplicity $m-1$, one has $\frac{g(z)}{g'(z)} = 0$, the function $\frac{g(z)}{g'(z)}$ is holomorphic in a neighbourhood of z by Riemann removable singularity theorem, and $f(z) = z$.

Step 2: To simplify the formulas, assume that $z = 0$. Since $g(0) = 0$ and g has a zero of multiplicity m, the Taylor decomposition for g in 0 takes form

$$g(z) = a_m z^m + a_{m+1} z^{m+1} + ...,$$

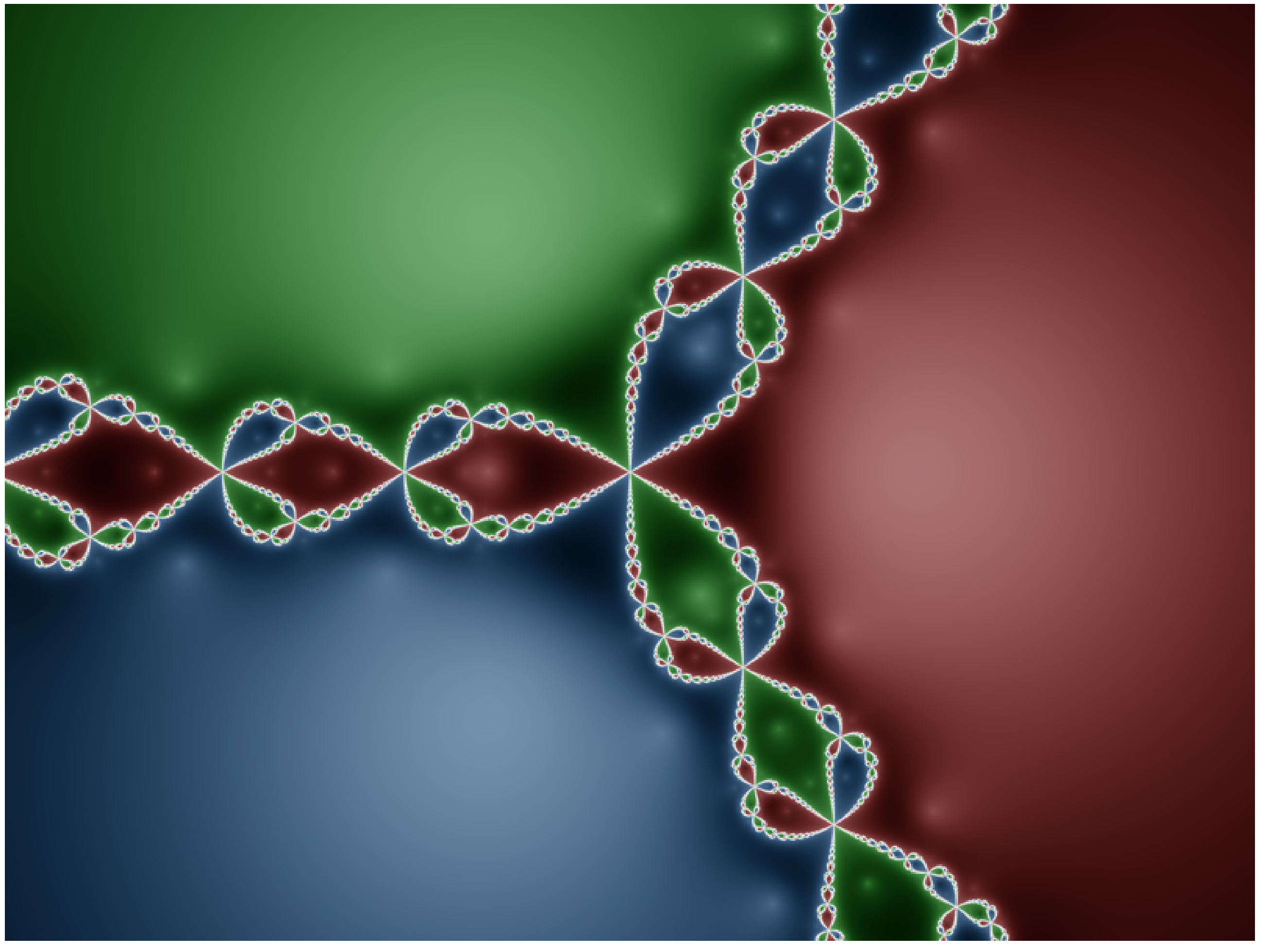
where $a_m \neq 0$. Then

$$g'(z) = m a_m z^{m-1} + (m+1) a_{m+1} z^m + ...$$

Let $u(z) = a_m + a_{m+1} z + a_{m+2} z^2 + ...$ and $v(z) = m a_m + (m+1) a_{m+1} z + (m+2) a_{m+2} z^2 + ...$ Since $\sum |a_i|$ converges, these series converge in an appropriate neighbourhood of zero. Clearly, $g(z) = z^m u(z)$ and $g'(z) = z^{m-1} u(z)$, which gives $\frac{g(z)}{g'(z)} = z \frac{u(z)}{v(z)}$, and $\frac{d}{dz} \frac{g(z)}{g'(z)}|_{z=0} = \frac{u(0)}{v(0)} = \frac{1}{m}$. This gives $f'(0) = 1 - \frac{1}{m}$. ■

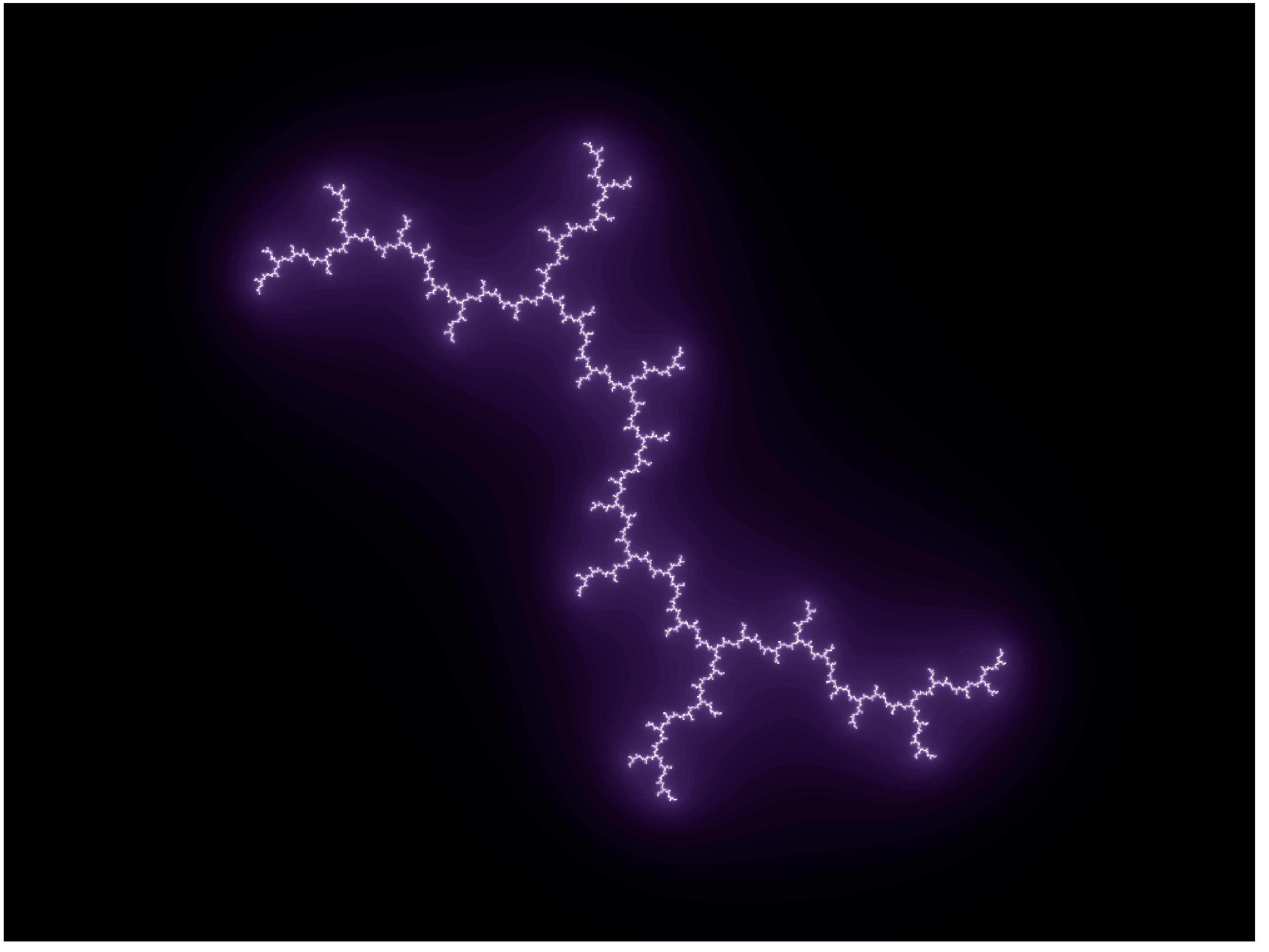
Fatou and Julia sets for $f(z) = \frac{1+2z^3}{3z^2}$

We apply the Newton iteration method to $g(z) = z^3 - 1$.



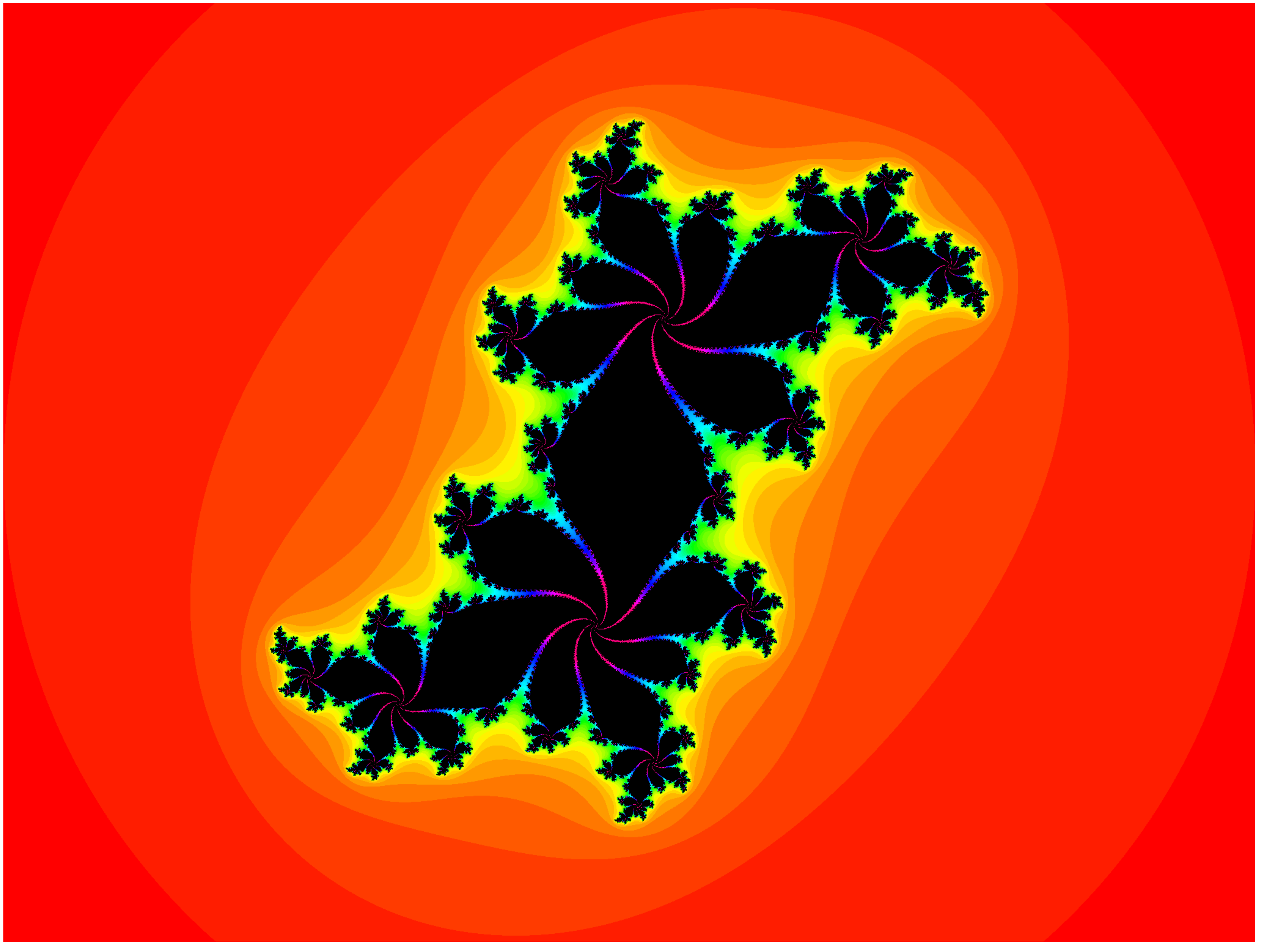
Julia set (in white) for the map $f(z) = \frac{1+2z^3}{3z^2} = z - \frac{g(z)}{g'(z)}$. Attractor basins for three roots of $g(z) = z^3 - 1$ are colored in red, green, blue.

Julia set for $f(z) = z^2 - \sqrt{-1}$

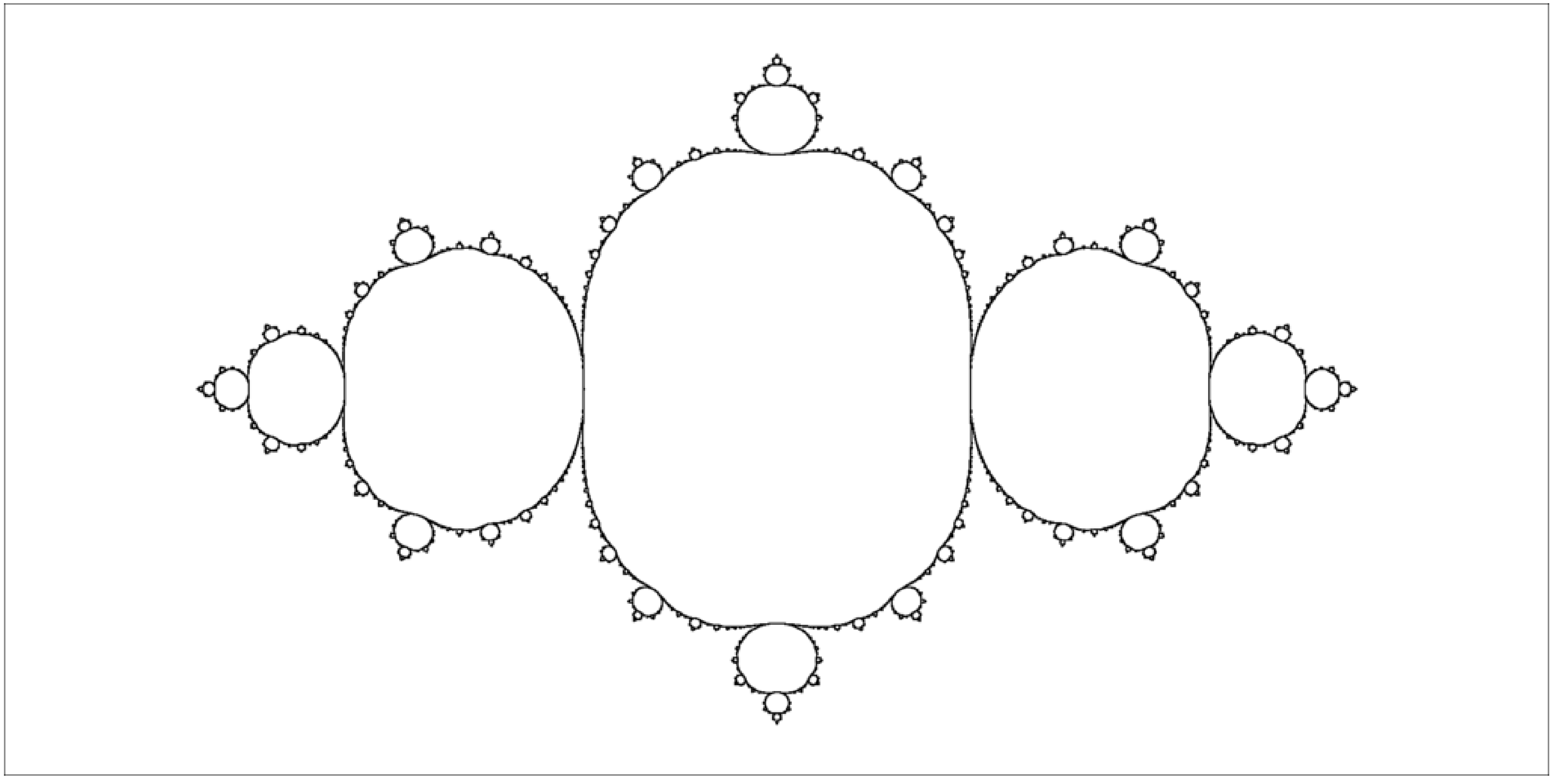


Julia set for $f(z) = z^2 - \sqrt{-1}$ is called **dendrite**.

Julia set for $f(z) = z^2 + 0.12 + 0.6\sqrt{-1}$



San Marco fractal

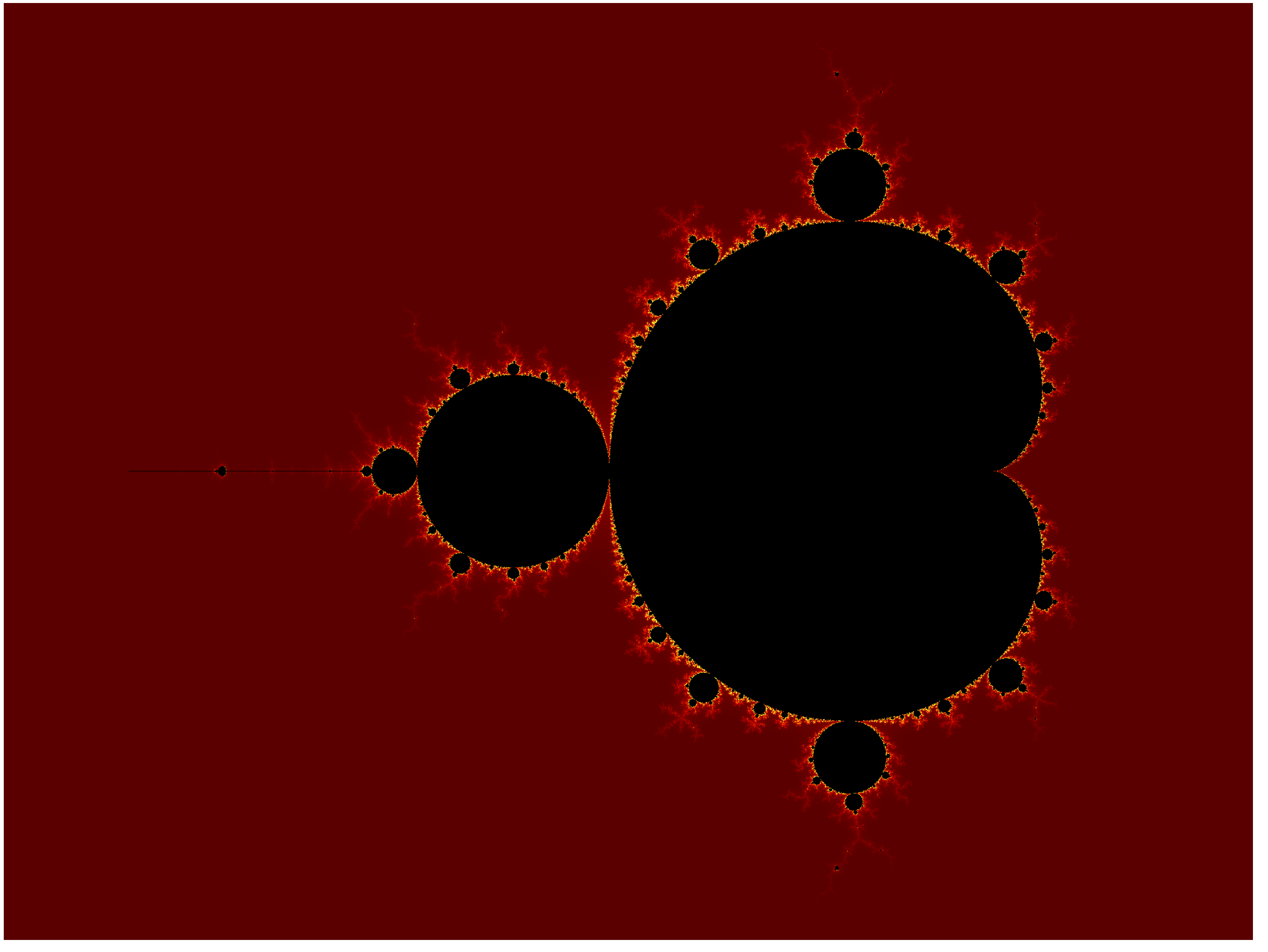


San Marco fractal is the Julia set for $f(z) = z^2 - 0.75$

St. Mark's Basilica, Venice

Mandelbrot set

DEFINITION: Mandelbrot set is the set of all c such that 0 belongs to the Fatou set of $f(z) = z^2 + c$.



Properties of Fatou and Julia sets

REMARK: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic map. **Then the Fatou $F(f)$ and Julia set $J(f)$ of f are f-invariant.**

LEMMA: (Iteration lemma) For each k, $J(f) = J(f^k)$, where f^k is k-th iteration of f.

Proof. Step 1: Clearly, $F(f^k) \subset F(f)$, because $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact when $\overline{\{f, f^2, f^3, ...\}}$ is compact.

Step 2: Conversely, suppose that $X = F(f^k)$; then $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact, but then $\overline{\{f, f^{k+1}, f^{2k+1}, f^{3k+1}, ...\}}$ is also compact as a continuous image of a compact (the composition is continuous in uniform topology), same for $\overline{\{f^2, f^{k+2}, f^{2k+2}, f^{3k+2}, ...\}}$, and so on. Then $\overline{\{f, f^2, f^3, ...\}}$ is obtained as a union of k compact sets. **Therefore, $F(f) \subset F(f^k)$.** ■

Properties of Fatou and Julia sets (2)

THEOREM: Julia set of polynomial map $f: \mathbb{C} \longrightarrow \mathbb{C}$ is non-empty, unless $\deg f \leqslant 1$.

Proof: Let $\Delta \subset \mathbb{C}P^1$, and $n(g)$ the number of critical points of a holomorphic function g in Δ. Then $n(g) = \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g'}{g} dz$, and this number is locally constant in uniform topology if g has no critical points on the boundary. Since the number of critical points of f^i is $i \deg f - 1$, it converges to infinity, hence f^i **cannot converge to a holomorphic function everywhere.** ■