Complex manifolds of dimension 1

lecture 13: Tilings and polyhedral hyperbolic manifolds

Misha Verbitsky

IMPA, sala 232

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Points of ramification (different proof)

DEFINITION: Let $\varphi: X \longrightarrow Y$ be a holomorphic map of complex manifolds, not constant on each connected component of X. Any point $x \in X$ where $d\varphi = 0$ is called **a ramification point** of φ. **Ramification index** of the point x is the number of preimages of $y' \in Y$, for y' in a sufficiently small neighbourhood of $y = \varphi(x)$.

THEOREM 1: Let X, Y be compact Riemannian surfaces, $\varphi: X \longrightarrow Y$ a holomorphic map, and $x \in X$ a ramification point. Then there is a neighbourhood of $x \in X$ biholomorphic to a disk Δ, **such that the map $\varphi|_\Delta$ is equivalent to $\varphi(x) = x^n$,** where n is the ramification index.

Proof: Take neighbourhoods $U \ni x$, $V \ni \varphi(x)$ which are biholomorphic to a disk, with $\varphi(U) \subset V$. Write the Taylor decomposition for φ in 0: $\varphi(x) = a_n x^n + a_{n+1} x^{n+1} + ... = x^n u(x)$, where $a_n \neq 0$. where $u(x) = a_n + a_{n+1} x + a_{n+1} x^2 + ...$ Since $u(x)$ is invertible, one can choose a branch $v(x) := \sqrt[n]{u(x)}$ in a neighbourhood of 0. Then $\varphi(x) = z^n$, where $z = xv(x)$. ■

Hyperelliptic curves and hyperelliptic equations (reminder)

REMARK: Let $M \xrightarrow{\Psi} N$ be a C^∞-map of compact smooth oriented manifolds. Recall that **degree** of Ψ is number of preimages of a regular value n, counted with orientation. Recall that **the number of preimages is independent from the choice of a regular value $n \in N$,** and **the degree is a homotopy invariant.**

DEFINITION: **Hyperelliptic curve** S is a compact Riemann surface admitting a holomorphic map $S \longrightarrow \mathbb{C}P^1$ of degree 2 and with $2n$ ramification points of degree 2.

DEFINITION: **Hyperelliptic equation** is an equation $P(t,y) = y^2 + F(t) = 0$, where $F \in \mathbb{C}[t]$ is a polynomial with no multiple roots.

DEFINITION: Let $P(t,y) = y^2 + F(t) = 0$ be a hyperelliptic equation. **Homogeneous hyperelliptic equation** is $P(x,y,z) = y^2 z^{n-2} + z^n F(x/z) = 0$, where $n = \deg F$.

REMARK: The set of solutions of $P(x,y,z) = 0$ is singular, but an algebraic variety of dimension 1 has a natural desingularization, called **normalization**. Define the involution $\tau(x,y,z) = (x,-y,z)$. Clearly, $\tau(S) = S$. **The involution τ is extended to the desingularization S,** giving $S/\tau = \mathbb{C}P^1$ because $\mathbb{C}P^1$ is the only smooth holomorpic compactification of $\mathbb{C}$ as we have seen already.

Hyperbolic polyhedral manifolds (reminder)

DEFINITION: A **polyhedral manifold** of dimension 2 is a piecewise smooth manifold obtained by gluing polygons along edges.

DEFINITION: Let $\{P_i\}$ be a set of polygons on the same hyperbolic plane, and M be a polyhedral manifold obtained by gluing these polygons. Assume that all edges which are glued have the same length, and we glue the edges of the same length. Then M is called **a hyperbolic polyhedral manifold**. We consider M as a metric space, with the path metric induced from P_i.

CLAIM: Let M be a hyperbolic polyhedral manifold. Then for each point $x \in M$ which is not a vertex, x **has a neighbourhood which is isometric to an open set of a hyperbolic plane.**

Proof: For interior points of M this is clear. When x belongs to an edge, it is obtained by gluing two polygons along isometric edges, hence the neighbourhood is locally isometric to the union of the same polygons in $\mathbb{H}^2$ aligned along the edge. ■

Hyperbolic polyhedral manifolds: interior angles of vertices (reminder)

DEFINITION: Let $v \in M$ be a vertex in a hyperbolic polyhedral manifold. **Interior angle** of v in M is sum of the adjacent angles of all polygons adjacent to v.

EXAMPLE 1: Let $M \longrightarrow \Delta$ be a ramified n-tuple cover of the Policate disk, given by solutions of $y^n = x$. We can lift split Δ to polygons and lift the hyperbolic metric to M, obtaining M as a union of n times as many polygons glued along the same edges. **Then the interior angle of the ramification point is $2\pi n$.**

EXAMPLE 2: Let $\Delta \longrightarrow M$ be a ramified n-tuple cover, obtained as a quotient $M = \Delta/G$, where $G = \mathbb{Z}/n\mathbb{Z}$. Split Δ onto fundamental domains of G, shaped like angles adjacent to 0. Then the quotient Δ/G gives an angle with its opposite sides glued. **It is a hyperbolic polyhedral manifold with interior angle $\frac{2\pi}{n}$ at its ramification point.**

EXAMPLE 3: Let D be a diameter bisecting a disk Δ, and passing through the origin 0 and $P \subset \Delta$ one of the halves. The (unique) edge of P is split onto two half-geodesics E_+ and E_- by the origin. **Gluing E_+ and E_-, we obtain a hyperbolic polyhedral manifold with a single vertex and the interior angle π.**

Sphere with n points of angle π (reminder)

EXAMPLE: Let P be a bounded convex polygon in $\mathbb{H}^2$, α the sum of its angles, and a_i, $i = 1, ..., n$ median points on its edges E_i. Each a_i splits E_i in two equal intervals. We glue them as in Example 3, and glue all vertices of P together. **This gives a sphere M with hyperbolic polyhedral metric,** one vertex ν with angle α (obtained by gluing all vertices of P together) and n vertices with angle π corresponding to $a_i \in E_i$.

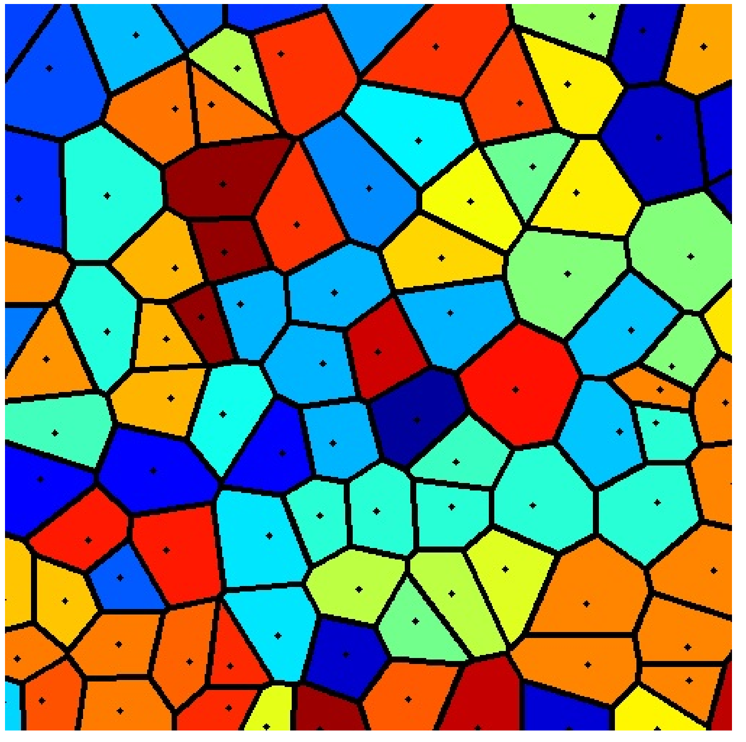
REMARK: Assume that $\alpha = 2\pi$, that is, M is isometric to a hyperbolic disk in a neighbourhood of ν. We equip M with a complex structure compatible with the hyperbolic metric outside of its singularities. A neighbourhood of each singularity is isometrically identified with a neighbourhood of 0 in Δ/G, where $G = \mathbb{Z}/2\mathbb{Z}$. We put a complex structure on Δ/G as in Example 2. **This puts a structure of a complex manifold on M.**

THEOREM: (Alexandre Ananin)
Let M be the hyperbolic polyhedral manifold obtained from the polygon P with n vertices as above. Assume that n is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**
Proof: later today.

REMARK: Clearly, **M_1 is hyperelliptic.**

Voronoi partitions

DEFINITION: Let M be a metric space, and $S \subset M$ a finite subset. **Voronoi cell** associated with $x_i \in S$ is $\{z \in M \ \ | \ d(z,x_i) \leqslant d(z,x_i) \forall j \neq i\}$. **Voronoi partition** is partition of M onto its Voronoi cells.



Voronoi partition

Fundamental domains and polygons

DEFINITION: Let Γ be a discrete group acting on a manifold M, and $U \subset M$ an open subset with piecewise smooth boundary. Assume that for any non-trivial $\gamma \in \Gamma$ one has $U \cap \gamma(U) = \emptyset$ and $\Gamma \cdot \overline{U} = M$, where $\overline{U}$ is closure of U. Then $\overline{U}$ is called **a fundamental domain** of the action of Γ.

THEOREM: Let Γ be a discrete group acting on a hyperbolic plane $\mathbb{H}^2$ by isometries. **Then Γ has a polyhedral fundamental domain P.** If, moreover, $\mathbb{H}^2/\Gamma$ has finite volume, ∂P has at most finitely many points on Abs.

Proof: Clearly, $\operatorname{Vol}(P) = \operatorname{Vol}(\mathbb{H}^2/\Gamma)$. This takes care of the last assertion, because polygons with infinitely many points on Abs have infinite volume

To obtain P, take a point $s \in \mathbb{H}$, and let P be the Voronoi cell associated with the set $\Gamma \cdot s$. ■

Discrete subgroups in $SO^+(1,2)$

LEMMA: Let $\Gamma \subset SO^+(1,2)$ be a discrete subgroup, and $S \subset \mathbb{H}^2$ the set of points with non-trivial stabilizer. **Then S is discrete**, and **the stabilizer of each point finite.**

Proof. Step 1: Since the center and the angle uniquely defines an elliptic isometry of $\mathbb{H}^2$, infinitely many angles of rotation around a given point give a non-discrete subset of $SO^+(1,2) = \operatorname{Iso}(\mathbb{H}^2)$. **This is why the stabilizer $\operatorname{St}_\Gamma(x)$ of each point x is finite.**

Step 2: Let now x_i be a sequence of fixed points converging to $x \in \mathbb{H}^2$. If the order of $\operatorname{St}_\Gamma(x_i)$ goes to infinity, **the limit point of the set $\bigcup_i \operatorname{St}_\Gamma(x_i)$ contains all rotations around x, hence Γ cannot be discrete.** If the order of $\operatorname{St}_\Gamma(x_i)$ is bounded, we can replace $\{x_i\}$ by a subsequence of points such that $\operatorname{St}_\Gamma(x_i)$ has order n for $n \in \mathbb{Z}^{\geqslant 2}$ fixed. Then the limit set of $\bigcup_i \operatorname{St}_\Gamma(x_i)$ **contains an elliptic rotation of order n around x,** a contradiction. ■

Group quotients and polyhedral hyperbolic manifolds

THEOREM: Let $\Gamma \subset SO^+(1,2)$ be a discrete subgroup, and $\mathbb{H}^2/\Gamma$ the quotient. **Then $\mathbb{H}^2/\Gamma$ is isometric to a polyhedral hyperbolic manifold.**

Proof. Step 1: First, **we prove that the quotient $\mathbb{H}^2/\Gamma$ is a manifold.** Indeed, outside of the set of fixed points, the action of Γ is properly discontinuous, and the quotient is smooth. For each fixed point p, it contains a neighbourhood where $\mathrm{St}_\Gamma(p)$ acts as a finite order rotation group $\mathbb{Z}/n\mathbb{Z}$ on a disk, and the quotient is smooth by Theorem 1.

Step 2: Let $\Gamma \cdot x$ be an orbit of Γ in $\mathbb{H}^2$. The Voronoi partition gives a polygonal fundamental domain P for the Γ-action. The space $\mathbb{H}^2/\Gamma$ is obtained by gluing the appropriate edges of P and then taking a quotient by appropriate finite groups stabilizing different points in P. Let $\mathrm{St}_\Gamma(P)$ be the stabilizer of P in Γ. Then $\mathbb{H}^2/\Gamma$ **is a quotient of a polyhedral hyperbolic manifold by $\mathrm{St}_\Gamma(P)$.**

Group quotients and polyhedral hyperbolic manifolds (2)

THEOREM: Let $\Gamma \subset SO^+(1,2)$ be a discrete subgroup, and $\mathbb{H}^2/\Gamma$ the quotient. **Then $\mathbb{H}^2/\Gamma$ is isometric to a polyhedral hyperbolic manifold.**

Proof. Step 1: First, **we prove that the quotient $\mathbb{H}^2/\Gamma$ is a manifold.**

Step 2: Let $\Gamma \cdot x$ be an orbit of Γ in $\mathbb{H}^2$. The Voronoi partition gives a polygonal fundamental domain P for the Γ action. Then **$\mathbb{H}^2/\Gamma$ is a quotient of a polyhedral hyperbolic manifold by $\mathrm{St}_\Gamma(P)$.**

Step 3: It remains to show that such a quotient is always a polyhedral hyperbolic manifold. Since Γ is discrete, the set of its fixed points is discrete, and the order of rotation is finite. For each interior point $x \in P$ with finite stabilizer, its stabilizer group acts on the set $\Gamma \cdot x$, hence it acts on P by isometries. Then we can cut P into isometric pieces, replacing P by a smaller fundamental domain with no fixed points in interior.

Since P is (strictly) convex, none of the vertices is fixed by a non-trivial element $\gamma \in \mathrm{St}_\Gamma(P)$. The only point which can be fixed by γ is the median m of the edge, but an isometry of the Voronoi tiling preserving m has order 2 and acts by rotations, hence cannot fix P. This means that $\mathrm{St}_\Gamma(P) = 0$ when there are no interior points of P preserved by $\gamma \in \Gamma$ ■

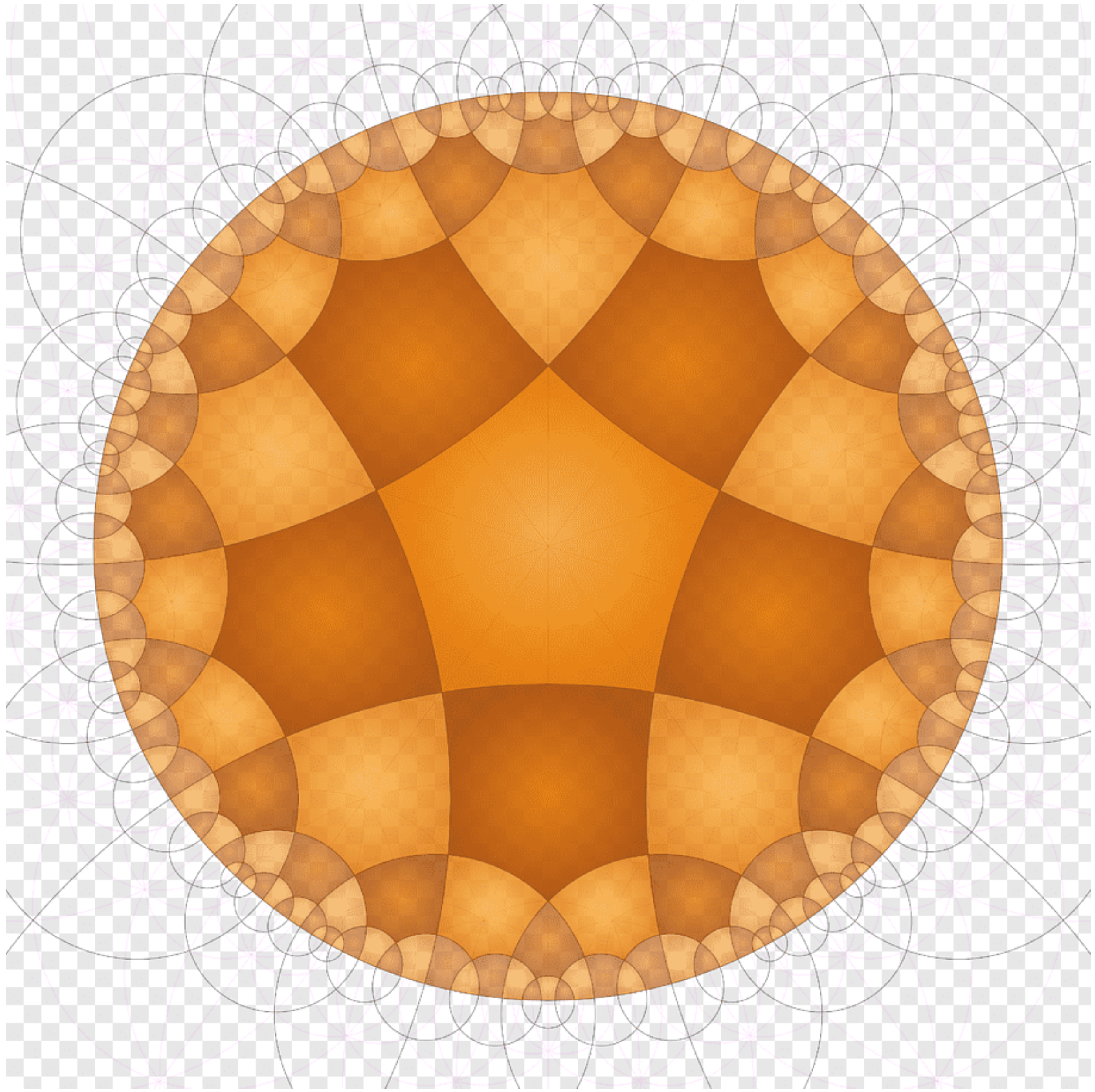
Semi-regular tilings

DEFINITION: A tiling of $\mathbb{H}^2$ is a partition of $\mathbb{H}^2$ onto polygons with finite volume. A tiling is **regular** if the group Γ of isometries preserving tilings acts transitively on vertices, edges and faces of the partition. A tiling T is **semi-regular** if Γ acts on the set of faces of T with finitely many orbits.

REMARK: Tilings is good a way to produce hyperbolic manifolds and Riemannian surfaces from a hyperbolic plane. Indeed, for any semi-regular tiling, T, **the quotient space $\mathbb{H}^2/\Gamma$ has finite volume.** Moreover, **$\mathbb{H}^2/\Gamma$ is compact if all polygons in T have no vertices in** Abs **(prove it)**.

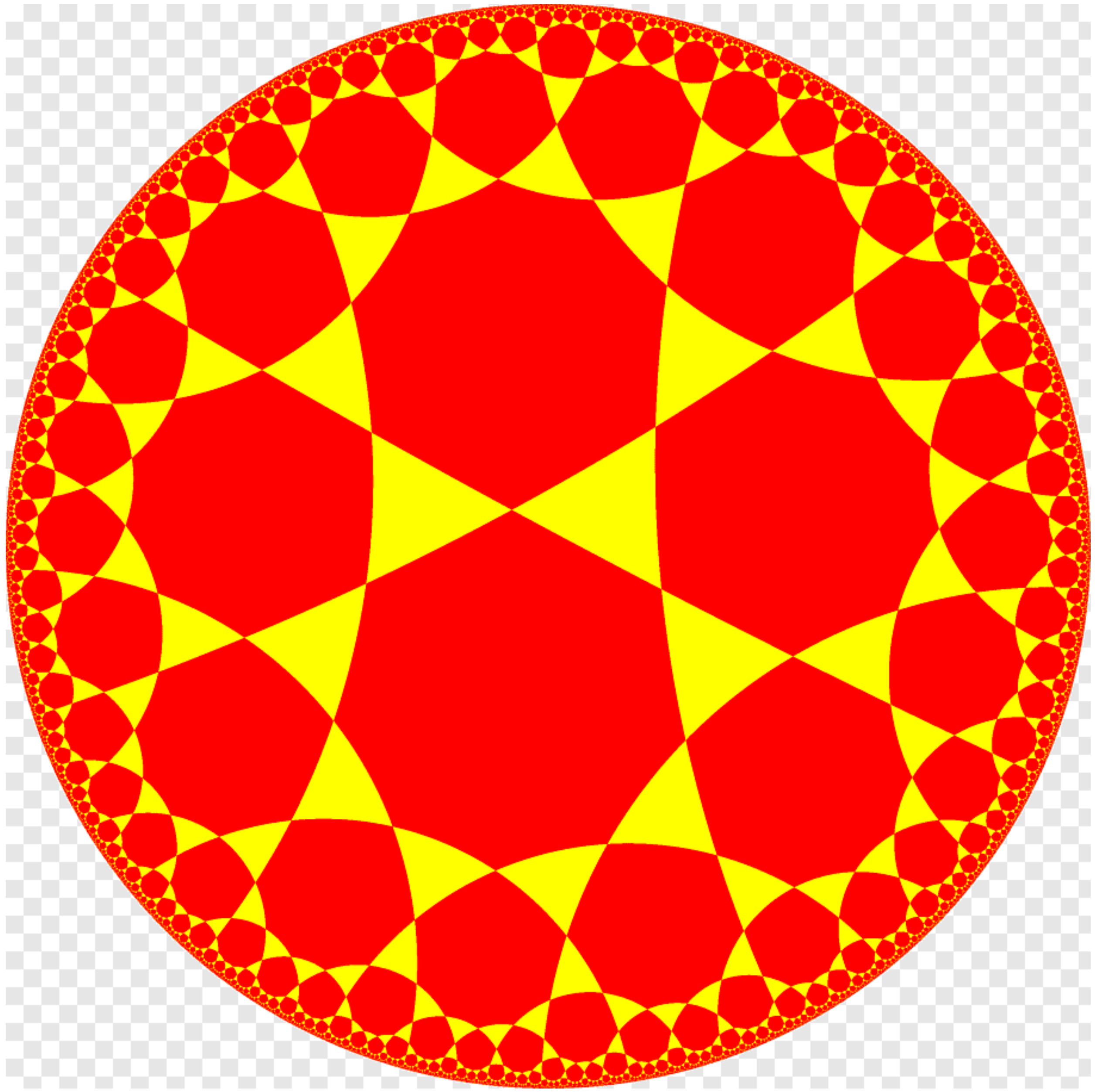
EXERCISE: Let T be a regular tiling of $\mathbb{H}^2$, and Γ the group of isometries of $\mathbb{H}^2$ preserving T. **Prove that any face of T is a fundamental domain for Γ.**

Regular tiling of $\mathbb{H}^2$ by right-angle pentagons



Regular tiling of $\mathbb{H}^2$ by right-angle pentagons

Semi-regular tiling of $\mathbb{H}^2$



Semi-regular tiling of $\mathbb{H}^2$ by octagons and triangles

Fundamental domains and tilings

REMARK: Bounded polygon in $\mathbb{H}^2$ is a polygon P such that $\overline{P}$ has no points in Abs, or, equivalently, such that the closure of P in $\mathbb{H}^2$ is compact.

CLAIM: Let T be a semi-regular tiling of $\mathbb{H}^2$ by bounded polygons. Then the group Γ of isometries of $\mathbb{H}^2$ preserving T **acts on $\mathbb{H}^2$ with a fundamental domain which is a bounded polygon.**

Proof: Let $\Gamma \cdot x$ be an orbit of $x \in \mathbb{H}^2$, and V_x the corresponding Voronoi domain. It would suffice to show that the closure of V_x is compact. Let $B_x(R) \subset \mathbb{H}^2$ be a a disc of radius R with center in x which contains a representative of each Γ-orbit on the tiles of T. There are finitely many orbits, and all tiles are compact, hence such a disk always exists. Then for every $y \in \mathbb{H}^2$, there exists $\gamma \in \Gamma$ such that $\gamma(y) \in B_x(R)$. Then $d(y, \gamma^{-1}(x)) \leqslant R$, hence either $y \in B_x(R)$ or $y \notin V_x$. We proved that $V_x \subset B_x(R)$, hence the Voronoi polygon is compact. ■

COROLLARY: Let Γ be a group of isometries of a semi-regular tiling. Then the quotient $\mathbb{H}^2/\Gamma$ **is a compact polyhedral hyperbolic manifold,** hence **is a compact Riemann surface.** ■

Cocompact subgroups of $PSL(2,\mathbb{R})$ without torsion

DEFINITION: A discrete subgroup $\Gamma \subset PSL(2,\mathbb{R})$ is **cocompact** if $\mathbb{H}^2/\Gamma$ is compact.

THEOREM: (a part of Poincaré uniformization theorem)
Let S be a compact Riemannian surface of genus > 1. **Then $S = \mathbb{H}^2/\Gamma$ for $\Gamma \subset PSL(2,\mathbb{R})$ freely acting on $\mathbb{H}^2$.**

Proof will be given later in these lectures, if time permits.

THEOREM: Let $\Gamma \subset PSL(2,\mathbb{R})$ be a discrete group. The action of Γ on $\mathbb{H}^2$ **is free if and only if it does not contain elliptic elements.** If, moreover, Γ is cocompact, **all its non-trivial elements are hyperbolic.**

Proof: The first assertion is clear, because **elliptic elements have fixed points on $\mathbb{H}^2$, hyperbolic and parabolic act without fixed points.**

To prove the second, let $\gamma \in \Gamma = \pi_1(S)$. Then corresponding class in $\pi_1(S)$ can be represented by a closed geodesic $s \subset S$ **(prove it).** Let $\tilde{s} \subset \mathbb{H}^2$ be its preimage. Since $\tilde{s}$ contains x and $\gamma(x)$, **the action of γ preserves the geodesic $\tilde{s}$,** hence γ is hyperbolic. ∎

The proof of Ananin Theorem

EXAMPLE: Let P be a bounded convex polygon in $\mathbb{H}^2$, α the sum of its angles, and a_i, $i = 1, ..., n$ median points on its edges E_i. Each a_i splits E_i in two equal intervals. We glue them as in Example 3, and glue all vertices of P together. **This gives a sphere M with hyperbolic polyhedral metric,** one vertex ν with angle α (obtained by gluing all vertices of P together) and n vertices with angle π corresponding to $a_i \in E_i$.

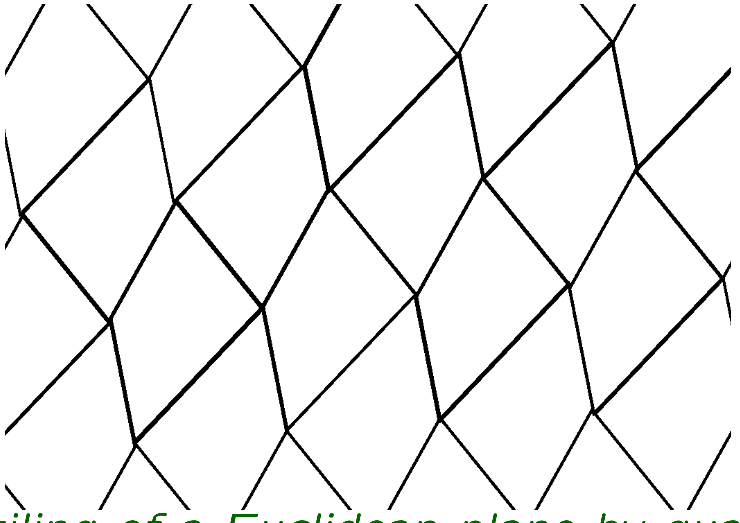
REMARK: Assume that $\alpha = 2\pi$, that is, M is isometric to a hyperbolic sphere around ν. We equip M with a complex structure compatible with the hyperbolic metric outside of its singularities. A neighbourhood of each singularity is isometrically identified with a neighbourhood of 0 in Δ/G, where $G = \mathbb{Z}/2\mathbb{Z}$.

THEOREM: (Alexandre Ananin)
Let M be the hyperbolic polyhedral manifold obtained from the polyhedron P as above. Assume that $m = n$ is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Strategy of a proof: We tile the hyperbolic plane $\mathbb{H}^2$ by copies of P. We show that the group Γ of oriented isometries of this tiling has a subgroup Γ_2 of index 2, freely acting on $\mathbb{H}^2$, such that $\mathbb{H}^2/\Gamma = M$, and $\mathbb{H}^2/\Gamma_2$ is its ramified covering with ramification in $a_1, ..., a_n$.

The proof of Ananin Theorem (2)

Proof. Step 1: Fix an isometric embedding $P \hookrightarrow \mathbb{H}^2$. Let Γ be the group of isometries generated by central symmetries (that is, rotations with angle π) around $a_1, ..., a_n$. Then the images of P tile $\mathbb{H}^2$ in such a way that P is a fundamental domain of Γ. Indeed, $\Gamma \cdot P$ covers the whole $\mathbb{H}$ (it is open and closed). However, rotating around the edge adjacent to a given vertex of P, we go through a full circle after adding all interior angles one by one. Since the sum of interior angles of P is 2π, we arrive back to P, hence **the images of P intersect with P only on the edge**. We proved that P is a fundamental domain of Γ, which is the isometry group of the tiling of $\mathbb{H}^2$ by copies of P.



Ananin tiling of a Euclidean plane by quadrangles

The proof of Ananin Theorem (3)

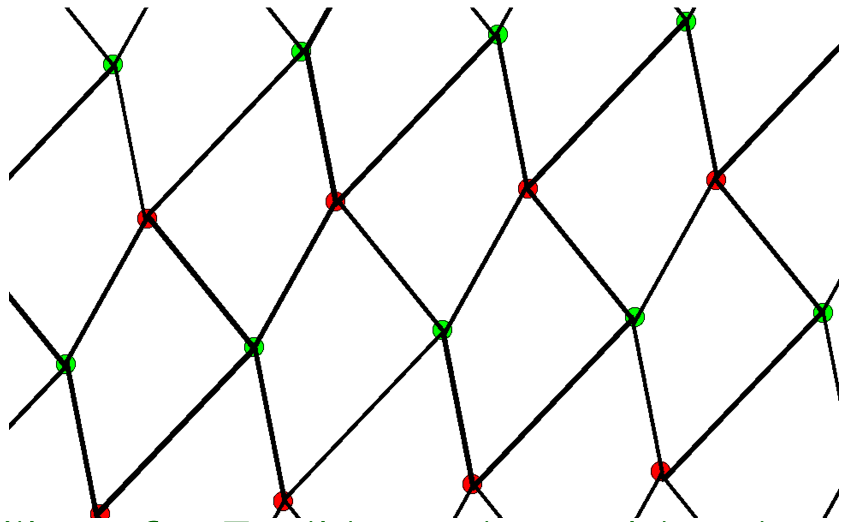
THEOREM: (Alexandre Ananin)
Let M be the hyperbolic polyhedral manifold obtained from the polyhedron P as above. Assume that n is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Proof. Step 1: Fix an isometric embedding $P \hookrightarrow \mathbb{H}^2$. Let Γ be the group of isometries generated by central symmetries around $a_1, ..., a_n$. **Then the images of P tile $\mathbb{H}^2$ in such a way that P is a fundamental domain of Γ.**

Step 2: Now we prove that $\mathbb{H}^2/\Gamma = M$. Indeed, Γ acts freely on P, and its action is non-free only in $a_1, ..., a_n$, which are fixed points of appropriate central symmetries. These central symmetries identify two opposite halves of each edge, hence $\mathbb{H}^2/\Gamma$ is obtained by gluing half of each edge of P with the opposite half.

The proof of Ananin Theorem (4)

Step 3: It remains to construct an index 2 subgroup $\Gamma_2 \subset \Gamma$ freely acting on $\mathbb{H}^2$. We color the vertices of the tiling constructed above in colors red and green in such a way that connected vertices have different colors. This is possible if P has even number of vertices.



Ananin tiling of a Euclidean plane with colored vertices

The central symmetries τ_j generating Γ exchange red and green vertices. Let $\Gamma_2 \subset \Gamma$ be a subgroup generated by products of even number of τ_j. Clearly, Γ_2 is a subgroup of all elements $\gamma \in \Gamma$ preserving colors of the vertices. Any element of Γ has to most 1 fixed point in the middle of an edge of a tile, **hence Γ_2 acts on $\mathbb{H}^2$ freely.**