Complex manifolds of dimension 1

lecture 14: Flow of diffeomorphisms

Misha Verbitsky

IMPA, sala 232

February 14, 2020

Vector fields and derivations

DEFINITION: Let M be a smooth manifold, and $X: C^\infty M \longrightarrow C^\infty M$ an operator. We call X **a vector field** if in each coordinate system $x_1, ..., x_n$ on M, the map X can be written as $X(f) = \sum \alpha_i \frac{df}{dx_i}$.

DEFINITION: A map X from a ring to itself is called **a derivation** it it satisfies **the Leibnitz rule**: $X(fg) = gX(f) + fX(g)$. Further on, we shall mostly consider derivations $X: C^\infty M \longrightarrow C^\infty M$. Such derivations are tacitly **assumed to be $\mathbb{R}$-linear.**

DEFINITION: **Support** $\operatorname{Supp}(f)$ of a continuous function f is the closure of the set of all points where it is not equal to 0. An operator $X: C^\infty M \longrightarrow C^\infty M$ is **local** if it maps a function with support in K to a function with support in K, for each $K \subset M$.

REMARK: Clearly, **all vector fields are local derivations.**

Derivations are local

THEOREM: Any derivation $X: C^\infty M \longrightarrow C^\infty M$ is local.

Proof. Step 1: Let $K, L \subset M$ be non-intersecting closed sets, f be a function with support in K, and g a function satisfying $g|_L = 1$ and $\operatorname{Supp}(g) \cap K = \emptyset$. Then $0 = X(fg) = gX(f) + fX(g)$. Restricted to L, **this gives $X(f)|_L = 0$,** because $g|_L = 1$ and $f|_L = 0$.

Step 2: Let us prove now that $\operatorname{Supp}(X(f)) \subset K = \operatorname{Supp}(f)$. Let $x \notin K$. We need to show that $x \notin \operatorname{Supp}(X(f))$.

Choosing an appropriate coordinate system on $M \backslash K$, we find a function g such that $\operatorname{Supp}(g) \subset M \backslash K$ and $g = 1$ in a neighbourhood V of x. Then $X(f)|_V = 0$ by Step 1. **This implies that $x \notin \operatorname{Supp}(X(f))$.** ■

Hadamard's Lemma

LEMMA: (Hadamard's Lemma)
Let f be a smooth function on $\mathbb{R}^n$, and x_i the coordinate functions. **Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty \mathbb{R}^n$.**

Proof: Let $t \in \mathbb{R}^n$. Consider a function $h(t) \in C^\infty \mathbb{R}^n$, $h(t) = f(tx)$. Using the chain rule, we get $\frac{dh}{dt} = \sum \frac{d}{dx_i} f(tx) x_i$, obtaining

$$f(x) - f(0) = \int_0^1 \frac{dh}{dt} dt = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx) dt.$$

■

COROLLARY: Let $\mathfrak{m}_0$ be an ideal of all smooth functions on $\mathbb{R}^n$ vanishing in 0. **Then $\mathfrak{m}_0$ is generated by coordinate functions.** ■

COROLLARY: Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $df\big|_{T_x(M)} = 0$. **Then $f \in \mathfrak{m}_x^2$.**

Proof: $f(x) = \sum_{i=1}^n x_i g_i(x)$, where all g_i vanish in 0. ■

EXERCISE 1: Let $X : C^\infty M \longrightarrow C^\infty M$ be a derivation, and $f \in \mathfrak{m}_x^2$. **Prove that $X(f) \in \mathfrak{m}_x$.**

Polynomial vector fields and derivations

LEMMA 1: Let k be any field, $k[x_1, ..., x_n]$ the ring of polynomials, and $A \supset k[x_1, ..., x_n]$ any ring. Then **any k-linear derivation $X: k[x_1, ..., x_n] \longrightarrow A$ is expressed as $X(F) = \sum \alpha_i \frac{df}{dx_i}$, where $\alpha_i = X(x_i)$.**

Proof: A derivation X on $\mathbb{R}[x_1, ..., x_n]$ is determined by Leibnitz formula and $X(x_1), ..., X(x_n)$. On monomials the Leibnitz formula gives

$$X(x_1^{a_1} x_2^{a_2} ... x_n^{a_n}) = a_1 X(x_1) x_1^{a_1 - 1} x_2^{a_2} ... x_n^{a_n} + a_2 X(x_2) x_1^{a_1} x_2^{a_2 - 1} ... x_n^{a_n} + ... \\ + a_n X(x_n) x_1^{a_1} x_2^{a_2 - 1} ... x_n^{a_n - 1}.$$

Therefore, for any polynomial $F \in \mathbb{R}[x_1, ..., x_n]$, we have $X(F) = \sum \alpha_i \frac{dF}{dx_i}$, where $\alpha_i = X(x_i)$. ■

Vector fields and derivations

THEOREM 1: Any derivation $X : C^\infty M \longrightarrow C^\infty M$ is a vector field on M.

Proof. Step 1: Since derivations are local, **it suffices to prove this statement on $\mathbb{R}^n$.**

Step 2: For polynomial functions $F \in \mathbb{R}[x_1, ..., x_n]$, we have $X(F) = \sum \alpha_i \frac{dF}{dx_i}$, where $\alpha_i = X(x_i)$ (Lemma 1). **Therefore, $X\Big|_{\mathbb{R}[x_1,...,x_n]}$ is a vector field.**

Step 3: Given a derivation δ, write $\delta_0 := \sum \alpha_i \frac{df}{dx_i}$, where $\alpha_i = \delta(x_i)$. To finish Theorem 1, it suffices to show that $\delta_1 := \delta - \delta_0 = 0$. **This would follow if we prove that all derivations δ_1 vanishing on all polynomial functions vanish.**

Step 4: By Hadamard's lemma, for each $x \in \mathbb{R}^n$, one has $f \in \mathfrak{m}_x^2$ modulo linear functions. Since δ_1 vanishes on linear functions, one has $\delta_1(C^\infty\mathbb{R}^n) = \delta_1(\mathfrak{m}_x^2)$ for each $x \in \mathbb{R}^n$ (Hadamard's lemma). However, $\delta_1(\mathfrak{m}_x^2) \in \mathfrak{m}_x$ because δ_1 is a derivation (Exercise 1). **Therefore, $\delta_1(f)$ vanishes everywhere for all $f \in C^\infty\mathbb{R}^n$.** ■

Flow of diffeomorphisms

DEFINITION: Let $V: M\times [a,b] \longrightarrow M$ be a smooth map such that for all $t\in [a,b]$ the restriction $V_t := V\big|_{M\times \{t\}}: M \longrightarrow M$ is a diffeomorphism. Then V is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f\in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c} f$. **Then $f\longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}\cdot V_t^* g + \frac{d}{dt}V_t^* f\cdot V_t^*(g)$ by the Leibnitz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f\cdot (V_t^{-1})^*\frac{d}{dt}V_t^* g + g\cdot (V_t^{-1})^*\frac{d}{dt}V_t^* f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t\in [a,b]$, and $V: M\times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^*\frac{d}{dt}V_t = v_t$ for each $t\in [a,b]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

Automorphisms of the ring of functions

REMARK: Each diffeomorphism $\psi: M \longrightarrow M$ induces an automorphism of the ring of smooth functions on M, $f \mapsto \psi^* f$.

THEOREM: Let M be a manifold. Then **any automorphism $\Psi: C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Proof. Step 1: Given a point $x \in M$, denote by I_x **the maximal ideal of x**, that is, the ideal of all functions vanishing in x. On a compact manifold, any maximal ideal is obtained this way. Indeed, if an ideal $I \subset C^\infty M$ has no common zeros, for each $y \in M$ there exists $f_y \in I$ which does not vanish in y. Denote by U_y the open set where $f_y \neq 0$. Then $\{U_y\}$ is an open cover of M. Finding a finite subcover, we obtain a finite number of functions $f_i \in I$ such that $\bigcap_i U_{f_i} = M$. Then **the function $\sum f_i^2 \in I$ is invertible, hence $I = C^\infty M$ is not a maximal ideal.** For non-compact manifolds, **points of M are the same as ideals $I \subset C^\infty M$ such that $C^\infty M/I = \mathbb{R}$ (prove it).**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi: M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Automorphisms of the ring of functions (2)

THEOREM: Let M be a compact manifold. Then **any automorphism $\Psi : C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi : M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Step 3: All open subsets of M can be obtained as unions of open sets $U_f := f^{-1}(\mathbb{R}\backslash 0)$, where $f \in C^\infty M$ **(prove it)**. However, $f(x) = 0$ if and only if $f \in I_x$. Then U_f can be considered as a set of maximal ideals I_x such that $f \notin I_x$. Since Ψ maps U_f to $U_{\Psi(f)}$, the corresponding map ψ is continuous on M. This implies that ψ **is a homeomorphism.**

Step 4: Finally, Ψ maps coordinate functions on $U \subset M$ to coordinate functions on $\psi^{-1}(U)$, hence this homeomorphism is smooth. ■

Solutions of ODE (1)

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$, and $V: M \times [0,a] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [0,a]$, and $V_0 = \mathrm{Id}$. Then V_t is called **an exponent of v_t**.

Theorem 1: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$. Then **the exponent of v_t is unique. It always exists** when v_t has compact support.

Solutions of ODE (2)

Theorem 2: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$. Then **the exponent of v_t is unique. It always exists** when v_t vanish (for all t) outside of a compact set $K \subset M$.

Proof: To construct a flow of diffeomorphisms $V_t = e^{v_t}$ it suffices to find a family of automorphisms $\Psi_t: C^\infty M \longrightarrow C^\infty M$ smoothly depending on $t \in [0,a]$ such that $\Psi_t^{-1}\frac{d}{dt}\Psi_t = v_t$. This is the same as to solve the ordinary differential equation

$$\frac{df_t}{dt} = v_t(f_t) \qquad (*)$$

for any given f_0. Then $\Psi_t(f_0) := f_t$ clearly satisfies $\frac{d}{dt}\Psi_t(f_0) = v_t\Psi_t(f_0)$.

To finish the proof, **we need to show that a solution of (*) exists and is unique,** and **to prove that Ψ_t defined this way is an automorphism, that is, satisfies $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$.**

Existence and uniqueness of solutions of ODE

THEOREM: Let v_t be a vector field on a manifold M. Consider the differential equation

$$\frac{dx_t}{dt} = v_t(x_t), \qquad (*)$$

where $x_t \in M$, and $t \in [0, a]$. Suppose that v_t has compact support. **Then (*) has a unique solution for each initial value x_0.**

Proof: Existence and uniqueness of solutions of (*) follows from Peano and Picard-Lindelöf theorem. Recall that a function $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **Lipschitz** if $|\mu(x) - \mu(y)| < C|x - y|$ for all x, y. Let D be an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f \in C^\infty D$, and

$$\frac{df_t}{dt} = v(t, f(t)) \qquad (**)$$

a continuous first-order differential equation defined on D. **(Peano) Then for every initial value f_0 there exists a solution of (**) defined on a small interval $[0, \varepsilon]$.** Moreover **(Picard-Lindelöf) the solution is unique if v is Lipschitz**. Notice that v is Lipschitz on any compact set if it is smooth. Finally, if there are functions $\alpha, \beta : [0, \infty[\longrightarrow [0, \infty[$ such that $|v_t(x)| < \alpha(t)|x| + \beta(t)$, **the solution exists globally for all $t \in [0, \infty[$.** ■

Derivations and automorphisms

To finish Theorem 2, **it would suffice to show that the map $f_0 \xrightarrow{\Psi_t} f_t$ obtained as a solution of $\frac{df_t}{dt} = v_t(f_t)$ is multiplicative:** $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$. From the definition of Ψ_t it follows

$$\frac{d}{dt}\Psi_t(fg) = v_t(f_t)g_t + f_t v(g_t)$$

and

$$\frac{d}{dt}\Big(\Psi_t(f)\Psi_t(g)\Big) = v_t(f_t)g_t + f_t v(g_t)$$

Therefore, both $\Psi_t(fg)$ and $\Psi_t(f)\Psi_t(g)$ are solition of a differential equation $\frac{d}{dt}(\chi_t) = v_t(\chi_t)$ with the same initial value $\chi_0 = fg$. **They are equal by uniqueness of solutions.**

The same argument proves the following lemma.

LEMMA: Let v, v' be commuting vector fields. **Then the corresponding diffeomorphisms commute.** Moreover, $V_t(v') = v'$, **where V_t is the diffeomorphism flow associated with v.**

Proof: Indeed, **exponents of commuting linear operators commute.** ■

Distributions

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let Π : $TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that $\Pi([x, y])$ **is** $C^\infty(M)$**-linear as a function of** x **and** y**.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenuus form vanishes.

Smooth submersions

DEFINITION: Let $\pi: M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi: M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: ("Ehresmann's fibration theorem")
Let $\pi: M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi: M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi: M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X,Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ■

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.