Complex manifolds of dimension 1

lecture 15: Frobenius theorem, sheaves, categories

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Flow of diffeomorphisms (reminder)

DEFINITION: Let $V: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $V_t := V\Big|_{M\times\{t\}}: M \longrightarrow M$ is a diffeomorphism. Then V is called **a flow of diffeomorphisms**.

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$, and $V: M \times [0,a] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [0,a]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

Theorem 1: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$. Then **the exponent of v_t is unique. It always exists** when v_t has compact support.

LEMMA: Let v, v' be commuting vector fields. **Then the corresponding diffeomorphisms commute.** Moreover, **$V_t(v') = v'$, where V_t is the diffeomorphism flow associated with v.**

Proof: Indeed, **exponents of commuting linear operators commute.** ■

Distributions (reminder)

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let Π : $TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that $\Pi([x, y])$ **is** $C^\infty(M)$**-linear as a function of** x **and** y**.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **holonomic**, or **involutive**, if its Frobenuus form vanishes.

Smooth submersions (reminder)

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and **$\pi|_U$ is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X, Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ∎

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \stackrel{\pi}{\longrightarrow} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Frobenius theorem: existence of integral submanifolds

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an interal submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is a projection to the leaf space of the distribution.

REMARK: When B is 1-dimensional (in this case one says that B **has rank 1**, denoted $\operatorname{rk} B = 1$), **Frobenius theorem follows from existence of the diffeomorphism flow associated with a vector field.** Indeed, locally we may assume that B admits a non-degenerate section v. Let $V_t : M \times \mathbb{R} \longrightarrow M$ be the corresponding flow of diffeomorphisms. Then $Z_m := V_t(\{m\} \times \mathbb{R})$ is tangent to v everywhere, hence it is a 1-dimensional manifold immersed in M. Clearly, Z_m is a leaf this distribution. **Since B is a tangent to a foliation, it is integrable.**

Further on we shall need the following exercise.

EXERCISE: Let $V_t = e^{v_t}$ be a diffeomorphism flow on M, and $F \subset TM$ a vector bundle. Assume that $[v_t, F] \subset F$. **Prove that then V_t preserves $F \subset TM$.**

Basic sub-bundles (1)

DEFINITION: Let $B \subset TM$ be an involutive sub-bundle. A sub-bundle $F \subset TM$ is called **basic** for B if $F \supset B$ and for all $b \in B, b' \in F$, one has $[b, b'] \in F$.

REMARK: One should think of basic sub-bundles as of **sub-bundles preserved by all diffeomorphisms obtaned from exponentiation of a vector field $v \in B$.**

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi : M \longrightarrow M_1$ projection to the leaf space of B, and $F \supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}(F_1) = F$.

Proof: Next slide.

Basic sub-bundles (2)

LEMMA: Let $B\subset TM$ be an integrable distribution, $\pi: M \longrightarrow M_1$ projection to the leaf space of B, and $F\supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1\subset TM_1$ such that $\pi^{-1}F_1=F$.

Proof. Step 1: Consider coordinates $x_1,...,x_n$ on M such that $x_{k+1} = \pi^*(x'_{k+1}),...,x_n=\pi^*(x'_n)$, where $x'_i, i=k+1,k+2,...,n$ are coordinates on M_1, and $\frac{d}{dx_1},...,\frac{d}{dx_k}$ generate B. Locally such coordinates always exist, because B is integrable. Denote by G a subgroup of Diff(M) obtained by exponents of $\frac{d}{dx_1},...,\frac{d}{dx_k}$. Since $[B,F]\subset F$, the corresponding diffeomorphisms preserve F. **Therefore, F is a G-invariant sub-bundle of TM.**

Step 2: Any G-invariant sub-bundle $F\supset B$ is obtained as $\pi^{-1}(F_1)$ for some sub-bundle $F_1\subset TM_1 = M/G$. Indeed, since the local action of G is free, the bundle F/B is locally generated over $C^\infty M$ by a basis $s_1,...s_k$ of G-invariant sections. We lift s_i to G-invariant sections $\overline{s}_i$ of F. The projection $M\longrightarrow M_1=M/G$ maps each $\overline{s}_i$ to a vector field $\tilde{s}_i\in M_1$. Since $\overline{s}_i$ are linearly independent, they form a basis in a sub-bundle $F_1\subset TM_1$ **(check this)**.

Step 3: Conversely, if F is lifted from $M_1=M/G$, if is G-invariant, hence $e^{tb}(b')\subset F$, and this gives $[b,b']\subset F$ **(check this)**. ■

Frobenius theorem (proof)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if it is **integrable**, that is, each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \stackrel{\pi}{\longrightarrow} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

Proof. Step 1: Consider a rank 1 sub-bundle $B_1 \subset B$. Using the diffeomorphism flow as above, we prove that B_1 is integrable. Since $[B_1, B] \subset B$, the bundle B is basic with respect to B_1. **Therefore, $B = \pi^{-1}(B')$ for some $B' \subset TM_1$, where M_1 is the leaf space of B_1.**

Step 2: Let $\pi : M \longrightarrow M_1$ be the projection to the leaf space. Then $B = \pi^{-1}(B')$, where $\operatorname{rk} B' = \operatorname{rk} B - 1$. Using induction in $\operatorname{rk} B$, we can assume that B' is integrable. Let $\pi_0 : M_1 \longrightarrow M_0$ be the projection to the leaf space of B', defined locally in M. **Then $\pi \circ \pi_0 : M \longrightarrow M_0$ is the projection to the leaf space of B.** ■

Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces (or abelian groups) $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to

$$\eta\Big|_{U_i \cap U_j} - \eta\Big|_{U_j \cap U_i}.$$

Vector fields as a sheaf

DEFINITION: Let M be a smooth manifold, and $X: C^\infty M \longrightarrow C^\infty M$ an operator. We call X **a vector field** if in each coordinate system $x_1, ..., x_n$ on M, the map X can be written as $X(f) = \sum \alpha_i \frac{df}{dx_i}$.

DEFINITION: A map X from a ring to itself is called **a derivation** it it satisfies **the Leibnitz rule**: $X(fg) = gX(f) + fX(g)$. Further on, we shall mostly consider derivations $X: C^\infty M \longrightarrow C^\infty M$. Such derivations are tacitly **assumed to be $\mathbb{R}$-linear.**

THEOREM: Any derivation $X: C^\infty M \longrightarrow C^\infty M$ is a vector field on M.

Proof: See Lecture 14.

EXAMPLE: Let M be a smooth manifold, and $U \longrightarrow \operatorname{Der}(C^\infty U)$ map an open set U to the space of all derivations on the ring $C^\infty U$. **Then** $U \longrightarrow \operatorname{Der}(C^\infty U)$ **is a sheaf,** called **the sheaf of vector fields**, or **tangent sheaf**. Indeed, a vector field $X(f) = \sum \alpha_i \frac{df}{dx_i}$ can be restricted to an open subset **(check the rest of sheaf axioms)**.

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

Sheaves of modules

REMARK: Let $A : \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\varphi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

EXAMPLE: Clearly, **tangent bundle is a vector bundle**.

Locally constant sheaves

DEFINITION: Let $\mathcal{F}$ be a sheaf on M which takes a connected non-empty open subset $U \subset M$ to a vector space or abelian group $\mathbb{V}$. Extend $\mathcal{F}$ to all open sets using the gluing axiom. Then $\mathcal{F}$ is called **the constant sheaf**, denoted $\mathbb{V}_M$.

EXERCISE: Prove that **the constant sheaf $\mathbb{V}_M$ exists, and is unique up to isomorphism.**

EXERCISE: Let W be an open set in M, and S_W its set of connected components. Prove that $\mathbb{V}_M(W) = \mathbb{V}^{|S_W|}$.

DEFINITION: A **locally constant sheaf**, or **local system** is a sheaf which is locally isomorphic to a constant sheaf.

EXAMPLE: Let $\pi: M' \longrightarrow M$ be a covering. Given $U \subset M$, let S_U be the set of connected components of $\pi^{-1}(U)$, and set $\mathcal{F}(U) = \mathbb{V}^{|S_W|}$. We are going to define the restriction map r as follows. For an open subset $W \subset U$, consider the map $S_W \longrightarrow S_U$ induced by the natural embedding $\pi^{-1}(W) \stackrel{j}{\hookrightarrow} \pi^{-1}(U)$. For each direct sum component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \in \operatorname{im} j$, let $r_u: \mathbb{V}_u \longrightarrow \mathbb{V}_{j(u)}$ be identity. For a component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \notin \operatorname{im} j$, we set $r_u = 0$. Then $r := \bigoplus_{u\in S_U} r_u: \bigoplus_{u\in S_U} \mathbb{V} \longrightarrow \bigoplus_{w \in S_W} \mathbb{V}$. **This defines a locally constant sheaf on M (prove it).**

Riemann-Hilbert correspondence

THEOREM: Let M be a connected manifold, $\mathcal{C}_1$ the category of representations of $\pi_1(M)$, and $\mathcal{C}_2$ the category of local systems. **Then the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are naturally equivalent.**

THEOREM: The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ **are naturally equivalent to the category of vector bundles on M equipped with flat connection.**

EXERCISE: Try to prove these two theorems. If unable, try to google "Riemann-Hilbert correspondence" and "local system".