Complex manifolds of dimension 1

lecture 16: Local systems

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Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{O}b(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}b(\mathcal{C})$, one has a set $\mathcal{M}or(X, Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}or(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}b(\mathcal{C})$ there exists a morphism $\mathrm{Id}_A \in \mathcal{M}or(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}or(X, Y)$, one has $\mathrm{Id}_x \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y, X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:
Category of sets: its morphisms are arbitrary maps.
Category of vector spaces: its morphisms are linear maps.
Categories of rings, groups, fields: morphisms are homomorphisms.
Category of topological spaces: morphisms are continuous maps.
Category of smooth manifolds: morphisms are smooth maps.

Functors

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A **covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following set of data.

1. **A map** $F: \mathcal{O}b(\mathcal{C}_1) \longrightarrow \mathcal{O}b(\mathcal{C}_2)$.

2. **A map** $F: \mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X), F(Y))$ **defined for any pair of objects** $X, Y \in \mathcal{O}b(\mathcal{C}_1)$**.**

These data define a functor if they are **compatible with compositions,** that is, satisfy $F(\varphi) \circ F(\psi) = F(\varphi \circ \psi)$ for any $\varphi \in \mathcal{M}or(X,Y)$ and $\psi \in \mathcal{M}or(Y,Z)$, and **map identity morphism to identity** morphism.

Example of functors

A "natural operation" on mathematical objects is usually a functor. Examples:

1. A map $X \longrightarrow 2^X$ from the set X to the set of all subsets of X is a functor from the category $\mathcal{S}ets$ of sets to itself.

2. A map $M \longrightarrow M^2$ mapping a topological space to its product with itself is a functor on topological spaces.

3. A map $V \longrightarrow V \oplus V$ is a functor on vector spaces; same for a map $V \longrightarrow V \otimes V$ or $V \longrightarrow (V \oplus V) \otimes V$.

4. **Identity functor** from any category to itself.

5. A map from topological spaces to $\mathcal{S}ets$, putting a topological space to the set of its connected components.

EXERCISE: Prove that it is a functor.

Equivalence of functors

DEFINITION: Let $X, Y \in \mathcal{Ob}(\mathcal{C})$ be objects of a category $\mathcal{C}$. A mprphism $\varphi \in \mathcal{Mor}(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{Mor}(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{Ob}(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X: F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{Mor}(X,Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle \Psi_X}\downarrow & & \downarrow{\scriptstyle \Psi_Y} \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) \end{array}$$

Equivalence of categories

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exists a functor $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that the compositions $G \circ F$ and $G \circ F$ are equivaleent to the identity functors $\operatorname{Id}_{\mathcal{C}_1}$, $\operatorname{Id}_{\mathcal{C}_2}$.

REMARK: It is possible to show that this is equivalent to the following conditions: F **defines a bijection on the set of isomorphism classes of objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a bijection**

$$\mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y)).$$

for each $X, Y \in \mathcal{O}b(\mathcal{C}_1)$.

REMARK: From the point of view of category theory, **equivalent categories are two instances of the same category** (even if the cardinality of corresponding sets of objects is different).

Locally constant sheaves (reminder)

DEFINITION: Let $\mathcal{F}$ be a sheaf on M which takes a connected non-empty open subset $U \subset M$ to a vector space or abelian group $\mathbb{V}$. Extend $\mathcal{F}$ to all open sets using the gluing axiom. Then $\mathcal{F}$ is called **the constant sheaf**, denoted $\mathbb{V}_M$.

EXERCISE: Prove that **the constant sheaf $\mathbb{V}_M$ exists, and is unique up to isomorphism.**

EXERCISE: Let W be an open set in M, and S_W its set of connected components. Prove that $\mathbb{V}_M(W) = \mathbb{V}^{|S_W|}$.

DEFINITION: A **locally constant sheaf** is a sheaf which is locally isomorphic to a constant sheaf.

EXAMPLE: Let $\pi: \ M' \longrightarrow M$ be a covering. Given $U \subset M$, let S_U be the set of connected components of $\pi^{-1}(U)$, and set $\mathcal{F}(U) = \mathbb{V}^{|S_W|}$. We are going to define the restriction map r as follows. For an open subset $W \subset U$, consider the map $S_W \longrightarrow S_U$ induced by the natural embedding $\pi^{-1}(W) \stackrel{j}{\hookrightarrow} \pi^{-1}(U)$. For each direct sum component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \in \operatorname{im} j$, let $r_u: \ \mathbb{V}_u \longrightarrow \mathbb{V}_{j(u)}$ be identity. For a component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \notin \operatorname{im} j$, we set $r_u = 0$. Then $r := \bigoplus_{u \in S_U} r_u: \ \bigoplus_{u \in S_U} \mathbb{V} \longrightarrow \bigoplus_{w \in S_W} \mathbb{V}$. **This defines a locally constant sheaf on M (prove it).**

Étalé space of a sheaf

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, and $U,V \supset x$ be two open set containing $x \in M$. Two sections $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ are called **equivalent in x** if there exists an open set $W \ni x$ such that $W \subset U \cap V$ and $f|_W = g|_W$. **A germ of a sheaf $\mathcal{F}$ in x** is a class of equivalence of sections of $\mathcal{F}$ in all open sets $U \ni x$ under this equivalence relation. **The stalk** of a sheaf $\mathcal{F}$ in x is the space $\mathcal{F}_x$ of all germs in x.

DEFINITION: Let $E(\mathcal{F})$ be the set of all stalks of a sheaf $\mathcal{F}$ in all points $x \in M$. A germ $f \in \mathcal{F}_m$ is called **a limit of a sequence of germs $f_i \in \mathcal{F}_{m_i}$** if $\lim_i m_i = m$ and there exists a section $\tilde f$ of $\mathcal{F}$ over $U \ni x$ such that almost all f_i are germs of $\tilde f$. The **étalé topology** on $E(\mathcal{F})$ is defined as follows: a subset $K \subset E(\mathcal{F})$ is **closed in étalé topology** if it contains all its limit points.

REMARK: Usually $E(\mathcal{F})$ **is non-Hausdorff**.

Étalé space of a constant sheaf

CLAIM: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold, and $x \in M$ a connected subset. **Then the space of germs of $\mathcal{F}$ in x is equal to $\mathbb{V}$.**

Proof: Since $\mathcal{F}$ is constant, the set of its sections on any connected open set is equal to $\mathbb{V}$. This gives a natural map $r_x := \mathcal{F}(U) \longrightarrow \mathbb{V}$: we restrict $f \in \mathcal{F}(U)$ to a connected component U_1 of U containing x, and obtain an element of $\mathbb{V}$. **Clearly, two sections f, g are equivalent in K if and only if $r_x(f) = r_x(g)$.** This identifies $\mathbb{V}$ with the set of equivalence classes of sections in x. ■

Corollary 1: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold. **Then the étalé space $E(\mathcal{F})$ of $\mathcal{F}$ is identified with $\mathbb{V}$ disconnected copies of M.**

Proof: Indeed, a sequence $f_i \in \mathcal{F}_{m_i}$ converges to f if $\lim_i m_i = m$ and $r_{m_i}(f_i) = r_m(f)$ for almost all i. ■

Local systems

DEFINITION: Category of coverings of M is category $\mathcal{C}$ with $\mathcal{Ob}(\mathcal{C})$ all coverings and morphisms continuous maps of coverings compatible with projections to M.

DEFINITION: Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be continuous maps. **Fibered product** $M_1 \times_M M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x,y) \in M_1 \times M_2 \mid \pi_1(x) = \pi_2(y)\}$, with induced topology.

EXERCISE: Prove that **a fibered product of coverings is a covering**.

DEFINITION: An abelian group structure on a covering $\pi_1 : M_1 \longrightarrow M$ is a morphism of coverings $\mu : M_1 \times_M M_1 \longrightarrow M_1$ together with a morphism $e : M \longrightarrow M_1$ from a trivial covering to M_1 such that μ defines a structure of an abelian group on the set $\pi_1^{-1}(x)$ for each $x \in M$, with $e(x)$ a unit in this group.

REMARK: If, in addition, we have a group homomorphism $\mathbb{R}^* \longrightarrow \operatorname{Aut}_M(M_1, M_1)$ which equips each $\pi_1^{-1}(x)$ with a structure of a vector space, we obtain **a structure of a vector space on a covering**.

DEFINITION: A local system is a covering with a structure of an abelian group or a vector space.

Étalé space of a locally constant sheaf

THEOREM: Let $\mathcal{F} = \mathbb{V}_M$ be a locally constant sheaf on a manifold. **Then its étalé space $E(\mathcal{F})$ is a covering of M.**

Proof: Immediately follows from Corollary 1. ■

THEOREM: Category of locally constant sheaves is equivalent to the category of local systems.

Proof: Let $\mathcal{F}$ be a locally constant sheaf, and $E(\mathcal{F})$ its etale space. Then $E(\mathcal{F})$ is a covering of M. The structure of vector space on germs defines the structure of vector space on $E(\mathcal{F})$. **This gives a functor from locally constant sheaves to local systems.**

Conversely, let $\pi: M_1 \longrightarrow M$ be a local system, and $\mathcal{F}(U)$ be the space of the sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. Then $\mathcal{F}(U)$ is a vector space. The correspondence $U \longrightarrow \mathcal{F}(U)$ gives a sheaf, which is clearly locally constant. ■

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: When $M = [0, a]$ is an interval, any bunlde B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f. Therefore, the equation $\nabla_{d/dt}(b) = 0$ is a first order ODE, and **it has a unique solution for any initial value $b_0 = b\big|_{\{0\}}$.**

Curvature

DEFINITION: Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}}\eta \wedge \nabla b$. Then the operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b$, hence **the curvature is a $C^\infty M$-linear operator. We shall consider the curvature B as a 2-form with values in $\operatorname{End} B$.** Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an $\operatorname{End}(B)$-valued form acts on $\Lambda^* M \otimes B$ as above.

DEFINITION: A connection is **flat** if its curvature vanishes.

Riemann-Hilbert correspondence

THEOREM: Let M be a connected manifold, $\mathcal{C}_1$ the category of representations of $\pi_1(M)$, and $\mathcal{C}_2$ the category of local systems. **Then the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are naturally equivalent.**

THEOREM: The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ **are naturally equivalent to the category of vector bundles on M equipped with flat connection.**

EXERCISE: Try to prove these two theorems. If unable, try to google "Riemann-Hilbert correspondence" and "local system".