Complex manifolds of dimension 1

lecture 18: Flat affine manifolds and Newlander-Nirenberg theorem

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IMPA, sala 232

February 21, 2020

Torsion

REMARK: "Connection on a manifold M" denotes a connection on the bundle TM or $\Lambda^1 M$. Such a connection **induces a connection on all its tensor powers $TM^{\otimes i} \otimes \Lambda^1 M^{\otimes j}$** as in Lecture 17.

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

The torsion of ∇ is a map $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$ defined as $\nabla \circ \mathsf{Alt} - d$, where $\mathsf{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK:

$$\begin{aligned} T_\nabla(f\eta) = & \mathsf{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ & = f\Big[\mathsf{Alt}(\nabla\eta) - d\eta\Big] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

Therefore T_∇ is linear.

Torsion and commutator of vector fields

REMARK: Cartan formula gives

$$\begin{aligned} T_\nabla(\eta)(X,Y) =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X,Y) \\ =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)). \end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \operatorname{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_\nabla(\eta)(X,Y) = \eta\Big(\nabla_X(Y) - \nabla_Y(X) - [X,Y]\Big).$$

Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$.

We have just proved

CLAIM: The tensor $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$ is dual to the torsion map $\nabla \circ \operatorname{Alt} - d : \Lambda^1 M \longrightarrow \Lambda^2 M$ defined above.

Flat affine manifolds

DEFINITION: Affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a composition of a linear map and a parallel translation.

DEFINITION: A **flat affine manifold** is a manifold M equipped with an atlas $\{U_i\}$ such that all transition maps are affine. In this case, U_i are called **affine charts**.

REMARK: Let M be a flat affine manifold, U an affine chart. Consider the basis in $\Lambda^1 U$ given by the coordinate 1-forms $dx_1, ..., dx_n$. Any affine map puts dx_i to a linear combination of coordinate 1-forms, hence the subsheaf in $\Lambda^1 M$ sheaf generated by dx_i is locally constant. **Riemann-Hilbert correspondence gives a natural flat connection $\nabla : \Lambda^1 M \longrightarrow \Lambda^1 M \otimes \Lambda^1 M$ such that $\nabla(dx_i) = 0$.**

THEOREM: Let M be a flat affine manifold, and ∇ a flat connection on M constructed above. **Then ∇ is torsion-free.** Moreover, **every torsion-free flat connection is obtained from a flat affine structure this way.**

Flat affine manifolds and torsion-free connections

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Proof. Step 1: Consider a bundle B over M trivialized by a frame $b_1, ..., b_n$. Then **there exists a unique connection ∇ such that $\nabla(b_i) = 0$.** Indeed, $\nabla\left(\sum_{i=1}^n f_i b_i\right) = \sum df_i \otimes b_i$.

Step 2: An affine structure gives a torsion-free flat connection as follows. Let $x_1, ..., x_n$ be flat affine coordinates on $U \subset M$. Then $dx_1, ..., dx_n$ is a frame trivializing $\Lambda^1 U$, and we can define a connection ∇ such that $\nabla(dx_i) = 0$ as in Step 1. Any affine transform maps the form dx_i to a linear combination of dx_i. Therefore, for any other set of coordinates $y_1, ..., y_n$ defining the same affine structure, one has $\nabla(dy_i) = 0$. This implies that **∇ is independent on the choice of coordinates.** It is flat because $\nabla^2(dx_i) = 0$ and torsion-free because $\operatorname{Alt}(\nabla\left(\sum_{i=1}^n f_i dx_i\right) = \sum df_i \wedge dx_i = d\left(\sum_{i=1}^n f_i dx_i\right)$.

Flat affine manifolds and torsion-free connections (2)

THEOREM: Let M be a flat affine manifold, and ∇ a flat connection on TM constructed above. **Then ∇ is torsion-free.** Moreover, **every torsion-free flat connection is obtained from a flat affine structure this way.**

Step 3: It remains to show that every torsion-free, flat connection ∇ on M is obtained this way. By Riemann-Hilbert correspondence (Lecture 17) in a neighbourhood of each point there exists a frame $b_1, ..., b_n \in \Lambda^1 M$ such that $\nabla(b_i) = 0$. Since $\operatorname{Alt}(\nabla b_i) = db_i = 0$, each form b_i is closed. Poincaré lemma implies that $b_i = dx_i$. Since the forms dx_i are linearly independent, the derivative of the map $\kappa(m) := (x_1(m), ..., x_n(m))$ is invertible. Then κ is locally a diffeomorphism to $\mathbb{R}^n$, and x_i are coordinates. Clearly, ∇ is a connection constructed from these coordinates as in Step 2. We obtained an atlas on M such that $\nabla(dx_i) = 0$ for each coordinate function x_i. **It remains only to show that this atlas defines a flat affine structure**.

Step 4: Clearly, $\nabla\left(\sum_{i=1}^n f_i dx_i\right) = 0$ if and only if all f_i are constant. The transition functions between coordinates map $m = (x_1, ..., x_n)$ to $y_i = \sum_{i=1}^n \varphi_i(m)$ such that $\nabla(dy_i) = 0$. Expressing each dy_i as $dy_i = \sum_{i=1}^n f_{ij} dx_i$ and using $0 = \nabla\left(\sum_{i=1}^n f_{ij} dx_i\right) = df_{ik} \otimes dx_i$, we obtain that all functions f_{ij} (partial derivatives of the transition functions $\varphi_i(m)$) are constant. A function with constant partial derivatives is always affine, hence **the transition functions between charts are affine.** ■

Newlander-Nirenberg theorem in dimension 1

The following theorem will be proven later today

THEOREM 1: Let (M, I) be a 1-dimensional almost complex manifold. **Then M locally admits a torsion-free, flat connection ∇ such that $\nabla(I) = 0$.**

This theorem immediately implies Newlander-Nirenberg in dimension 1.

THEOREM: Let (M, I) be a 1-dimensional almost complex manifold. **Then I is integrable.**

Proof. Step 1: Integrability of I means for each $m \in M$ there exists a neighbourhood U with coordinates x, y such that $x + \sqrt{-1}\, y$ is holomorphic with respect to I. Equivalently, this means that $dx + \sqrt{-1}\, dy \in \Lambda^{1,0}(M)$. **Equivalently, this means that $dy = I(dx)$.**

Step 2: To obtain such a coordinate system, take any x such that $\nabla dx = 0$ as above. Then $\nabla(Idx) = 0$ because $\nabla(I) = 0$. Since ∇ is torsion-free, this implies that $I(dx)$ is closed, hence $I(dx) = dy$ for some y. By Step 1, this implies that **the complex coordinate $x + \sqrt{-1}\, y$ is holomorphic with respect to I.** ■

Hodge decomposition on $\Lambda^2(M)$

Fix an almost complex manifold (M, I). Let $\Lambda^1_{\mathbb{C}}(M) := \Lambda^1(M) \otimes_{\mathbb{R}} \mathbb{C}$, and $\Lambda^1_{\mathbb{C}}(M) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$ be the Hodge decomposition.

EXERCISE: Prove that **the multiplicative map $\Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \xrightarrow{\mathsf{Alt}} \Lambda^2(M)$ is injective.**

DEFINITION: We denote the image of this map by $\Lambda^{1,1}(M)$. **The Hodge decomposition** on $\Lambda^2(M)$ is written as $\Lambda^2_{\mathbb{C}}(M) = \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M) \oplus \Lambda^{0,2}(M)$, where $\Lambda^{2,0}(M) = \Lambda^{1,0}(M) \wedge \Lambda^{1,0}(M)$ and $\Lambda^{0,2}(M) = \Lambda^{0,1}(M) \wedge \Lambda^{0,1}(M)$

Torsion-free connections on $\Lambda^{1,0}(M)$

DEFINITION: Let (M,I) be an almost complex manifold, and

$$\nabla: \Lambda^{1,0}M \longrightarrow \Lambda^1 M \otimes \Lambda^{1,0}M$$

a connection on $\Lambda^{1,0}$. It is called **torsion-free** if $\operatorname{Alt}(\nabla(\eta)) = d\eta$ for any $(0,1)$-form η.

CLAIM: Let $\nabla: \Lambda^{1,0}M \longrightarrow \Lambda^1 M \otimes \Lambda^{1,0}M$ be a torsion-free connection on $\Lambda^{1,0}$ Define a connection on $\Lambda^{0,1}M$ as $\nabla_X(\overline{\eta}) = \overline{\nabla_X(\eta)}$. This defines a connection $\tilde{\nabla}$ on $\Lambda^{1,0}M \oplus \Lambda^{0,1}M = \Lambda^1_{\mathbb{C}}M$. **Then ∇ is torsion-free if and only if $\tilde{\nabla}$ is torsion-free**, and, moreover, $\tilde{\nabla}(I) = 0$ **and any torsion-free connection on $\Lambda^1(M)$ preserving I is obtained this way.**

Proof: It is clear from construction that $\tilde{\nabla}$ is torsion-free and preserves the Hodge decomposition, hence satisfies $\tilde{\nabla}(I) = 0$. On the other hand, restriction of any torsion-free connection $\tilde{\nabla}$ to $\Lambda^{1,0}M$ is torsion-free, and can be used to recover the connection on $\Lambda^1 M$ because $\operatorname{Re}(\Lambda^{1,0}M) = \Lambda^1 M$. ■

Hodge decomposition and connections

REMARK: Let (B, ∇) be a complex bundle with connection. We decompose $\nabla : B \longrightarrow B \otimes \Lambda^1(M)$ onto a sum of its Hodge components, $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{1,0} : B \longrightarrow B \otimes \Lambda^{1,0}(M)$ and $\nabla^{0,1} : B \longrightarrow B \otimes \Lambda^{0,1}(M)$.

REMARK: Let ∇ be a torsion-free connection on an almost complex manifold (M, I) such that $\nabla(I) = 0$, and let $d^{0,1} : \Lambda^{1,0} \longrightarrow \Lambda^{1,1}(M)$ be the Hodge component of de Rham differential. Since ∇ is torsion-free, **the following diagram is commutative**

$$\begin{array}{ccc} \Lambda^{1,0} & \xrightarrow{\nabla^{0,1}} & \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \\ & \searrow^{d^{0,1}} & \downarrow \mathrm{Alt} \\ & & \Lambda^{1,1}(M) \end{array}$$

Since $\Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \xrightarrow{\mathrm{Alt}} \Lambda^{1,1}(M)$ is an isomorphism, **the Hodge component $\nabla^{0,1} : \Lambda^{1,0}M \longrightarrow \Lambda^{1,0} \otimes \Lambda^{0,1}(M)$ is uniquely determined by a complex structure.**

Existence of torsion-free connections

COROLLARY: Let ∇_1 and ∇_2 be torsion-free connections on a 1-dimensional almost complex manifold (M,I) such that $\nabla(I)=0$. **Then $\nabla_1 - \nabla_2|_{\Lambda^{1,0}}$ is a linear map $A: \Lambda^{1,0}M \longrightarrow \Lambda^{1,0}M \otimes \Lambda^{1,0}M$.** Conversely, **for any linear map $A: \Lambda^{1,0}M \longrightarrow \Lambda^{1,0}M \otimes \Lambda^{1,0}M$, the connection $\nabla_1 + A$ is torsion-free.**

Proof: The first statement is immediately implied by the previous corollary. The second statement is clear, because the multiplication $\Lambda^{1,0}M \otimes \Lambda^{1,0}(M) \longrightarrow \Lambda^{2,0}(M)$ vanishes. Indeed, $\Lambda^{1,0}M$ is 1-dimensional, hence $\Lambda^{2,0}(M) = 0$. ■

COROLLARY: Let (M,I) be a 1-dimensional almost complex manifold, and $\nabla: \Lambda^{1,0}M \longrightarrow \Lambda^1(M) \otimes \Lambda^{1,0}M$ a connection. Consider the map $d^{0,1}: \Lambda^{1,0}M \longrightarrow \Lambda^{0,1}(M) \otimes \Lambda^{1,0}M = \Lambda^{1,1}M$ as above. **This defines a connection $\nabla^{0,1} \oplus d^{0,1}$ which is torsion-free.**

Proof: By the same argument as above, **the relation $\operatorname{Alt}(\nabla(\eta)) = d\eta$ is equivalent to $\operatorname{Alt}(\nabla^{0,1}(\eta)) = d^{0,1}\eta$ for any $\eta \in \Lambda^{1,0}M$.** ■

COROLLARY: Let $\nabla: \Lambda^{1,0}M \longrightarrow \Lambda^1(M) \otimes \Lambda^{1,0}M$ be a torsion-free connection on $\Lambda^{1,0}M$, and $A: \Lambda^{1,0}M \longrightarrow \Lambda^{1,0}M \otimes \Lambda^{1,0}M$ a linear map. Then the curvature of the torsion-free connection $\nabla + A$ can be expressed as $(\nabla + A)^2 = [d^{0,1}, A]$. ■

Existence of torsion-free, flat connections

Notice that the bundles $\Lambda^{1,0}M$ and $\Lambda^{1,0}M \otimes \Lambda^{1,0}M$ are 1-dimensional. Therefore, $\operatorname{Hom}(\Lambda^{1,0}M, \Lambda^{1,0}M \otimes \Lambda^{1,0}M) = \Lambda^{1,0}M$ and **we can consider A as a (1,0)-form.** Under this identification the map $A \longrightarrow [d^{0,1}, A]$ is expressed as $A \longrightarrow d^{0,1}(A)$. This gives the following corollary,

COROLLARY: Let (M, I) be an almost complex manifold of dimension 1. Assume that the map $d^{0,1}: \Lambda^{1,0}M \longrightarrow \Lambda^2 M$ is surjective. **Then (M, I) admits a torsion-free flat connection.**

Proof: Choose a torsion-free connection ∇, and let $d^{0,1}(A) = \nabla^2$. **Then $\nabla - A$ is torsion-free and flat.** ■

Therefore, **the following theorem finishes the proof of Newlander-Nirenberg in dimension 1.**

THEOREM: Let (M, I) be an almost complex manifold of dimension 1. **Then every point has a neighbourhood U such that $d^{0,1}: \Lambda^{1,0}U \longrightarrow \Lambda^2 U$ is surjective to a space of 2-forms which can be extended continuously to the boundary of U.**

REMARK: This result is a special case of the **"Grothendieck-Dolbeault-Poincaré lemma"**.

Elliptic operators of second order

DEFINITION: Let M be an oriented n-manifold, and $D: C^\infty M \longrightarrow C^\infty M$ be a differential operator of second order, written in local coordinates as $D(f) = \sum_{i,j} a_{ij} \frac{d^2 f}{dx_i dx_j} + \sum_i b_i \frac{df}{dx_i} + c$, where the form a_{ij} is positive definite. Then D is called **an elliptic operator of second order**.

EXERCISE: Check that **this definition is independent from the choice of coordinates.**

I will use the following theorem without a proof

THEOREM: Let $D: C^\infty M \longrightarrow C^\infty M$ be an elliptic operator of second order on M. Then every point $x \in M$ has a neighbourhood U with compact closure and smooth boundary such that **D is surjective to a space of 2-forms which can be extended continuously to the boundary of U.**

Lie derivatives (reminder)

DEFINITION: Let $X \in TM$, and $i_X : \Lambda^i(M) \longrightarrow \Lambda^{i-1}(M)$ denote the **convolution map** $i_x(\eta)(\cdot, ..., \cdot) = \eta(X, \cdot, ..., \cdot)$.

DEFINITION: Let X be a vector field, e^{tX} the corresponding diffeomorphism flow, and Φ any tensor. Denote by $\operatorname{Lie}_X(\Phi)$ the tensor $\frac{d}{dt}|_{t=0}(e^{tX})^*(\Phi)$. This operation is called **the Lie derivative**.

REMARK: Clearly, for any function f, **the derivative $X(f)$ is equal to $\operatorname{Lie}_X f$.**

CLAIM: For any vector fields $X, Y \in TM$ **one has $\operatorname{Lie}_X Y = [X, Y]$.**

CLAIM: (Cartan's magic formula) $\operatorname{Lie}_X = di_X + i_X d$.

CLAIM: $[\operatorname{Lie}_X, i_Y] = i_{\operatorname{Lie}_X Y} = i_{[X,Y]}$.

Pluri-Laplacian on an almost complex 1-manifold

DEFINITION: Let (M,I) be an almost complex manifold. **The pluri-Laplacian** is a map $f \longrightarrow dd^c f$, where $d^c = IdI^{-1}$. It takes a function and maps it to a 2-form.

CLAIM: Let X be a vector field and $Y = I(X)$. Then

$$i_X i_Y(dd^c f) = -i_X i_Y dIdf = -i_X \operatorname{Lie}_Y Idf + i_X d i_Y Idf =$$
$$- i_{[Y,X]} Idf + \operatorname{Lie}_Y i_X Idf + i_X d i_Y Idf = -\operatorname{Lie}_{I[X,Y]} f + (\operatorname{Lie}_Y)^2 f + (\operatorname{Lie}_X)^2 f.$$

DEFINITION: Let M be a real 2-manifold, and $\omega \in \Lambda^2(M)$ a volume form. Since $\Lambda^2(M)$ is a line bundle, all its sections are proportional, and for any $\varphi \in \Lambda^2(M)$, one has $\varphi = f\omega$. **We write this as $f = \frac{\varphi}{\omega}$.**

COROLLARY: Let (M,I) be an almost complex 2-manifold, and $\omega \in \Lambda^2(M)$ a volume form. **Then the operator $f \longrightarrow \frac{dd^c f}{\omega}$ is elliptic.**

Proof: The operator $f \longrightarrow (\operatorname{Lie}_Y)^2 f + (\operatorname{Lie}_X)^2 f$ is clearly elliptic, and the correction term $\operatorname{Lie}_{I[X,Y]} f$ is first order. ■

Pluri-Laplacian and solutions of equation $d^{0,1}(\eta) = v$

To finish the proof of Newlander-Nirenberg it remains to prove the theorem

THEOREM: Let (M, I) be an almost complex manifold of dimension 1. **Then every point has a neighbourhood U such that $d^{0,1}: \Lambda^{1,0}U \longrightarrow \Lambda^2 U$ is surjective to a space of 2-forms which can be extended continuously to the boundary of U.**

Proof: Clearly, in complex dimension 1, the Hodge decomposition gives $d^{1,0} + d^{0,1} = d$. Therefore, $d^{1,0} = \frac{d + \sqrt{-1}\, d^c}{2}$ and $d^{0,1} = \frac{d - \sqrt{-1}\, d^c}{2}$. Then $dd^c = 2\sqrt{-1}\, d^{0,1} d^{1,0}$. **The operator $dd^c: C^\infty U \longrightarrow \Lambda^2 U$ is locally surjective because it is elliptic (see above).** Then $d^{0,1}: \Lambda^{1,0}U \longrightarrow \Lambda^2 U$ is also locally surjective. ■