Class test 1

Rules: This is a test assignment to do in class (instead of a lecture) for Wednesday, January 29, 2020. Please write the solutions and give to your monitor at the end of the class. You can refer to any theorems you like, but please give a precise reference and the full statement of the result you use.

Exercise 1.1. Let f be a holomorphic function on a disk Δ. Prove that the zero set Z_f of f is discrete in Δ.

Exercise 1.2. Let $P(t)$ be a polynomial. Prove that $f(z) = z - \frac{P(z)}{P'(z)}$ is holomorphic in a neighbourhood of any α which is a root of $P(t)$. Prove that $f(\alpha) = \alpha$ and $|f'(\alpha)| < 1$.

Exercise 1.3. Let $f(z)$ be a non-constant holomorphic function on a disk Δ. Prove that there exist $t, s \in]0, 1]$ such that the function $f(tz) - f(sz)$ has no zeros when $|z| = 1$.

Exercise 1.4. Prove that any holomorphic map from $\mathbb{C} \backslash 0$ to a disk Δ is constant.

Exercise 1.5. Prove that any holomorphic function on a disk Δ which is continuous on its boundary $\partial \Delta$ and takes real values on $\partial \Delta$ is constant, or find a counterexample.