Class test 2

Rules: This is a test assignment to do in class (instead of a lecture) for Friday, February 7, 2020. Please write the solutions and give to me at the end of the class. You can refer to any theorems you like, but please give a precise reference and the full statement of the result you use.

Exercise 2.1. Let $S \subset \mathbb{C} = \mathbb{R}^2$ be a closed 1-dimensional smooth submanifold, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function. Assume that f is holomorphic on $\mathbb{C} \backslash S$. Prove that f is holomorphic on $\mathbb{C}$.

Exercise 2.2. Let $f : \partial\Delta \longrightarrow \mathbb{C}$ be a continuous function. Prove that f can be extended to a holomorphic function on Δ, or find a counterexample.

Exercise 2.3. Let q be a quadratic form of signature (1,1) on $\mathbb{R}^2$ with integer coefficients. Prove that there is always a non-trivial rational pair $v = (a, b) \in \mathbb{R}^2$ such that $q(v) = 0$, or find a counterexample.

Exercise 2.4. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integer coefficients. Prove that there is always a non-trivial rational triple $u = (a, b, c) \in \mathbb{R}^3$ such that $q(u) = 0$, or find a counterexample.

Definition 2.1. Let $V = \mathbb{R}^3$ be vector space with quadratic form q of signature (1,2). A line[1] l in V is called **positive** if $q(x, x) > 0$ for some $x \in l$, **negative** if $q(x, x) < 0$ for some $x \in l$, and **isotropic** if $q(x, x) = 0$ for all $x \in l$. Let $\alpha \in SO^+(1, 2)$ be a non-trivial element. It is called **elliptic** if it preserves a positive line $l \in V$, **hyperbolic** if it preserves a negative line, and **parabolic** if all lines preserved by α are isotropic.

Exercise 2.5. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integer coefficients. Assume that there is no non-trivial rational triples $u = (a, b, c) \in \mathbb{R}^3$ such that $q(u) = 0$. Prove that $SO^+(1, 2)$ contains no parabolic elements with rational coefficients.

Exercise 2.6. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integer coefficients, $h \in SO^+(1, 2)$ a hyperbolic element with integer coefficients, and $P_h(t)$ its characteristic polynomial. Prove that $P_h(t)$ has precisely 1 rational root.

[1] Here, "line" means a 1-dimensional vector subspace.