Home assignment 1: holomorphic functions

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Exercise 1.1. Let f be a holomorphic function on a disk Δ. Prove that the zero set Z_f of f is discrete in Δ.

Exercise 1.2. Let $P(t)$ be a polynomial. Prove that $f(z) = z - \frac{P(z)}{P'(z)}$ is holomorphic in a neighbourhood of any α which is a root of $P(t)$. Prove that $f(\alpha) = \alpha$ and $|f'(\alpha)| < 1$.

Exercise 1.3. Let $f(z)$ be a non-constant holomorphic function on a disk Δ. Prove that there exist $t, s \in]0,1]$ such that the function $f(tz) - f(sz)$ has no zeros when $|z| = 1$.

Exercise 1.4. Let f be a holomorphic function on a unit disk Δ, continuous on $\partial\Delta$. Prove that $\frac{1}{2\pi}\int_\Delta f \operatorname{Vol}_{S^1} = f(0)$, where Vol_{S^1} is the Riemannian volume measure on a circle.

Exercise 1.5. Prove that any holomorphic map from $\mathbb{C}\backslash 0$ to a disk Δ is constant.

Exercise 1.6. Prove that any holomorphic function on a disk Δ which is continuous on its boundary $\partial\Delta$ and takes real values on $\partial\Delta$ is constant, or find a counterexample.