Home assignment 2: quadratic forms

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Exercise 2.1. Let q be a quadratic form of signature (1,1) on $\mathbb{R}^2$ with integer coefficients. Prove that there is always a non-trivial rational pair $v = (a, b) \in \mathbb{R}^2$ such that $q(v) = 0$, or find a counterexample.

Definition 2.1. The group $O(p, q)$ is the group of linear isometries of the $(p+q)$-dimensional vector space with scalar product of signature (p, q), and $SO(p, q) \subset O(p, q)$ is the group of isometries preserving the orientation. We use the notation $SO^+(p, q)$ for the connected component of $SO(p, q)$.

Exercise 2.2. Prove that $O(1, 1)$ has 4 connected components, and $SO(1, 1)$ has 2 connected components.

Exercise 2.3. Prove that $O(p, q)$ has 4 connected components, when $p, q > 0$, and $SO(p, q)$ has 2 connected components.

Hint. Use the previous exercise.

Exercise 2.4. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integral coefficients. Prove that there is always a non-trivial rational triple $u = (a, b, c) \in \mathbb{R}^3$ such that $q(u) = 0$, or find a counterexample.

Definition 2.2. Let $V = \mathbb{R}^3$ be vector space with quadratic form q of signature (1,2). A line[1] l in V is called **positive** if $q(x, x) > 0$ for some $x \in l$, **negative** if $q(x, x) < 0$ for some $x \in l$, and **isotropic** if $q(x, x) = 0$ for all $x \in l$. Let $\alpha \in SO^+(1, 2)$ be a non-trivial element. It is called **elliptic** if it preserves a positive line $l \in V$, **hyperbolic** if it preserves a negative line and has infinite order, and **parabolic** if all lines preserved by α are isotropic.

Exercise 2.5. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integral coefficients, $h \in SO^+(1, 2)$ a hyperbolic isometry with integral coefficients, and $P_h(t)$ its characteristic polynomial. Prove that $P_h(t)$ has precisely 1 rational root.

Exercise 2.6. Let $f: \partial\Delta \longrightarrow \mathbb{C}$ be a continuous function. Prove that f can be extended to a holomorphic function on Δ, or find a counterexample.

[1] Here, "line" means a 1-dimensional vector subspace.