# Home assignment 3: Lie groups

**Rules:** This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

**Definition 3.1. A Lie group** is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group $G$ **acts on a manifold** $M$ if the group action is given by the smooth map $G \times M \longrightarrow M$.

**Exercise 3.1.** Prove that $SL(n, \mathbb{R})$ is a Lie group. Prove that it is connected.

**Exercise 3.2.** Prove that the special unitary group $SU(n)$ acts transitively on the projective space $\mathbb{C}P^{n-1}$. Find the stabilizer $\operatorname{St}_x(SU(n))$ of a point $x \in \mathbb{C}P^{n-1}$. Prove that it is connected, or find a counterexample.

**Definition 3.2.** Let $W$ be an $n$-dimensional complex vector space equipped with a complex-linear non-degenerate quadratic form $s$. Consider **the complex orthogonal group** $O(n, \mathbb{C})$ of all matrices $A \in GL(W)$ preserving $s$. A subspace $V \subset W$ is called **isotropic** if $s\Big|_W = 0$. It is called **maximally isotropic**, or **Lagrangian**, if $\dim V = [n/2]$.

**Exercise 3.3.** Prove that $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ is a Lie group which has index 2 in $O(n, \mathbb{C})$. Prove that it is connected.

**Exercise 3.4.** Let $X$ be the space of all maximally isotropic suspaces in $W$ ("the maximally isotropic Grassmannian").

a. Prove that $O(n, \mathbb{C})$ acts on $X$ transitively.

b. Prove that $X$ is disconnected for $n = 2$.

c. Prove that it is connected for $n \geqslant 3$, or find a counterexample.