Home assignment 4: Orientation

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Definition 4.1. Determinant bundle on an n-dimensional smooth manifold is the bundle $\Lambda^n(M)$ of antisymmetric n-linear forms. A manifold is called **orientable** if its determinant bundle is trivial.

Exercise 4.1. Let S be a compact Riemann surface. Prove that there exists a free action of the group $Z/2$ on S reverting the orientation.

a. When S is a torus or a sphere.

b. When the genus of S is even.

c. When the genus of S is odd.

Exercise 4.2. Let M be an almost complex manifold. Prove that it is orientable.

Exercise 4.3. Let $S \subset \mathbb{R}^3$ be a smooth compact 2-dimensional submanifold of $\mathbb{R}^3$. Prove that it is orientable.

Exercise 4.4. Let M be a manifold with the fundamental group isomorphic to $\mathbb{Q}$. Prove that M is orientable.

Exercise 4.5. Let G be a Lie group. Prove that its fundamental group is commutative.

Exercise 4.6. Prove that an non-orientable compact 2-dimensional manifold does not admit a pseudo-Riemannian metric of signature (1,1), or find a counterexample.